84

Reports of the Department of Geodetic Science
Report No. 208

# CLOSED COVARIANCE EXPRESSIONS FOR GRAVITY ANOMALIES, GEOID UNDULATIONS, AND DEFLECTIONS OF THE VERTICAL IMPLIED BY ANOMALY DEGREE VARIANCE MODELS.

by

C.C. Tscherning

and

Richard H. Rapp

The Ohio State University
Department of Geodetic Science
1958 Neil Avenue
Columbus, Ohio 43210

May, 1974

Reports of the Department of Geodetic Science

Report No. 208

# CLOSED COVARIANCE EXPRESSIONS FOR GRAVITY ANOMALIES, GEOID UNDULATIONS, AND DEFLECTIONS OF THE VERTICAL IMPLIED BY ANOMALY DEGREE VARIANCE MODELS.

by

C.C. Tscherning
and
Richard H. Rapp

The Ohio State University
Department of Geodetic Science
1958 Neil Avenue
Columbus, Ohio 43210

May, 1974

ABSTRACT

This report first develops a new anomaly degree variance model by considering potential coefficient information to degree 20, and updated values of the point anomaly variance (1795 mgal$^2$), the 1° block variance (920 mgal$^2$) and the 5° block variance (302 mgal$^2$), the variances being given with respect to the Geodetic Reference System 1967. This new model was computed assuming that anomaly information was given on a sphere of radius 6371 km with the radius of the best fitting Bjerhammer sphere found to be 6369.8 km.

This new model and several other models were used to develop closed expressions for the covariance and cross-covariance functions between gravity anomalies, geoid undulations (or height anomalies), and deflections of the vertical. It is shown how these global covariance expressions can be modified for use as local covariances and for use when mean anomalies are being considered. A Fortran subroutine is provided for the determination of the covariance values implied by the recommended anomaly degree variance model.

## FOREWORD

This report was prepared by C.C. Tscherning, Research Associate, Department of Geodetic Science, and the Danish Geodetic Institute, and by Richard H. Rapp, Professor, Department of Geodetic Science. This work was sponsored, in part, by the Air Force Cambridge Research Laboratories, Bedford, Massachusetts, under Air Force Contract No. F19628-72-C-0120, The Ohio State University Research Foundation Project No. 3368B1, and in part by the Danish Geodetic Institute. The Air Force Technical Monitor is Mr. Bela Szabo.

Funds for the support of certain computations made in this study were made available through the Instruction and Research Computer Center of the Ohio State University.

The reproduction and distribution of this report was carried out through funds supplied by the Department of Geodetic Science. This report was also distributed by the Air Force Cambridge Research Laboratories as document AFCRL-TR-74-0231 (Scientific Report No. 14 under contract No. F19628-72-C-0120).

TABLE OF CONTENTS

## 1. Introduction

In carrying out the estimation of gravimetric dependent quantities using the methods of least squares collocation (Moritz, 1972) we need to have an analytical function that can be used to determine the covariance functions for such quantities as anomalies, deflections of the vertical, geoid undulations etc. Generally speaking a numerical covariance function for anomalies can be determined from anomaly data. The resultant function can be considered by determining a model for the anomaly degree variances. Tscherning (1972) has shown how such anomaly degree variance models may be used to determine the covariance models for several gravimetric quantities. Since we need the best estimates of our covariance models for the application of least squares collocation, it is appropriate that we use the latest available data in determining our models. In addition we are now at a stage where refinements in anomaly degree variance modeling, beyond that used by Rapp (1973) can be considered.

The purpose of this report is to describe recent computations made and subsequent analytical work that leads to improved analytical covariance models.

## 2. Preliminary Equations

In this section some of the relevant formulas to be used in later sections will be presented.

We first consider our covariance function which for the purposes of this report will be considered as stationary and isotropic. Then we can follow the standard definition (Heiskanen and Moritz, 1967) of the anomaly covariance as the mean product (at a given distance) of the anomaly pair $\Delta g_P$, $\Delta g_Q$. Thus:

$$C(P, Q) = \operatorname{cov}(\Delta g_P, \Delta g_Q) = M(\Delta g_P, \Delta g_Q) \tag{1}$$

On a plane the distance, or anomaly separation is usually specified by some linear distance (such as 20 km). If we deal with data on a sphere we usually considered the distance to be defined as $\psi$ a spherical arc so that we are interested in values of $C(\psi)$. At $\psi=0$, $C(\psi)$ becomes the anomaly variance. For the estimation of $C(\psi)$ from anomaly data given on the surface of a sphere, we can write (Heiskanen and Moritz, 1967, p. 258):

$$C(\psi) = \frac{1}{4\pi}\int_{\lambda=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{2\pi}\int_{\alpha=0}^{2\pi}\Delta g(\theta,\lambda)\,\Delta g(\theta',\lambda')\sin\theta\,d\theta\,d\lambda\,d\alpha \tag{2}$$

where $\theta$ is a polar angle (0 at the north pole), $\lambda$ is the longitude and $\alpha$ is an azimuth.

We will obtain from (2), a point anomaly covariance function if the $\Delta g$ values are point anomalies or we will obtain a mean anomaly covariance function (for a specific block size) if the $\Delta g$ values are mean anomalies. In practice the sphere is not completely covered by anomalies so that an expression that may be used to compute the covariance between any two functions $f_j$ and $f_k$ given in blocks on the sphere whose area is $A_j$ and $A_k$ respectively may be written: (Kaula, 1966a, p. I. B. 7).

$$C(\psi) = \frac{\sum A_j A_k f_j f_k}{\sum A_j A_k} \tag{3}$$

In our case $f_j = \overline{\Delta g}\,(\theta, \lambda)$ and $f_k = \overline{\Delta g}\,(\theta', \lambda')$ where the overbar signifies a mean anomaly. If the anomalies are given in equal area block (3) becomes:

$$C(\psi) = \frac{\sum f_j f_k}{n} \tag{4}$$

where n is the number of products taken at a given spherical distance $\psi$. In practice the distance $\psi$ to which a special product at distance $\psi_{jk}$ is determined by the equation:

$$\psi - \frac{\Delta\psi}{2} < \psi_{jk} < \psi + \frac{\Delta\psi}{2} \tag{5}$$

where $\Delta\psi$ is a suitably chosen range. In our numerical results to be discussed later, $\Delta\psi$ was specified to be $1^\circ$.

A more fundamental covariance function than that of the gravity anomalies is that of the disturbing potential, K(P, Q). We generally do not estimate K(P, Q) from numerical data, but rather consider the following series representation for it: (Moritz, 1972, p. 88):

$$K(P, Q) = \sum_{\ell=0}^{\infty} \sigma_\ell \left(\frac{R^2}{rr'}\right)^{\ell+1} P_\ell(\cos\psi) \tag{6}$$

where: $\sigma_\ell$ are the degree variances of the anomalous potential;
R is the radius of the Bjerhammar sphere;
$r, r'$ are the geocentric radii to points P and Q which are separated by a spherical radius $\psi$.

For convenience we let:

$$s = \frac{R^2}{rr'} \tag{7}$$

In the case that we are dealing with information at the approximate surface of the earth, it is convenient to take $rr' = R_e^2$ where $R_e$ is a mean earth radius. Then:

$$s = \left(\frac{R}{R_e}\right)^2 \tag{8}$$

We then can write:

$$K(P, Q) = \sum_{\ell=0}^{\infty} \sigma_\ell \, s^{\ell+1} P_\ell(\cos\psi) \tag{9}$$

We can also write the anomaly covariances in a series expression as (Moritz, 1972, p. 89):

$$C(P, Q) = \sum_{\ell=0}^{\infty} c_\ell \, s^{\ell+2} P_\ell(\cos\psi) \tag{10}$$

where $c_\ell$ are the anomaly degree variances. As written, equation (10) would yield a point anomaly covariance. In order to obtain a mean anomaly covariance we can use the $\beta_\ell$ functions of Meissl (1970, p. 23) or the $q_\ell$ functions of Pellinen (1966). Using $\beta_\ell$, the modification of equation (10) yields:

$$\overline{C}(P, Q) = \sum_{\ell=0}^{\infty} \beta_\ell^2 \, c_\ell \, s^{\ell+2} P_\ell(\cos\psi) \tag{11}$$

where P and Q now refer to anomaly blocks. $\beta_\ell$ is defined as follows: (Meissl, 1970, p. 24):

$$\beta_\ell = \frac{1}{1-\cos\psi_0} \quad \frac{1}{2\ell+1} \left[P_{\ell-1}(\cos\psi_0) - P_{\ell+1}(\cos\psi_0)\right] \tag{12}$$

where $\psi_0$ is the circular cap radius of the mean anomaly block whose covariance is to be computed. We have (for example):

$$\beta_0 = 1 \tag{12A}$$

$$\beta_1 = \tfrac{1}{2} \sin\psi_0 \cot\frac{\psi_0}{2} \tag{12B}$$

Since we usually deal with rectangular blocks of dimension $s^\circ$, the corresponding $\psi_0^\circ$ can be found simply by equating the areas of the circular cap and the square blocks. Assuming a plane figure we write (for small blocks only):

$$\psi_o^o = s^o/\sqrt{\pi} = 0.564s^o \tag{13}$$

As $\psi_o \to 0$, $\beta_\ell \to 1$

Since gravity anomalies are related to the disturbing potential by the following equation (valid in a spherical approximation which is the case considered here):

$$\Delta g = \frac{-\partial T}{\partial r} - \frac{2}{R} T \tag{14}$$

where T is the disturbing potential, we can relate the anomaly degree variances ($c_\ell$) and the degree variances of the anomalous potential ($\sigma_\ell$) by:

$$\sigma_\ell = \frac{R^2}{(\ell-1)^2} c_\ell \tag{15}$$

Analytic models for either $\sigma_\ell$, or $c_\ell$ have been described by Lauritzen (1973), Tscherning (1972), by Rapp (1973a) and implicitly by Kaula (1966b, p. 98).

The inverse of equation (10) is:

$$c_\ell = \frac{2\ell+1}{2} s^{-(\ell+2)} \int_0^\pi C(\psi) P_\ell(\cos\psi) \sin\psi \, d\psi \tag{16}$$

Equation (16) is written assuming $C(\psi)$ is a point anomaly covariance function referring to a sphere whose radius is $R_e$. If $C(\psi)$ is a point anomaly covariance function, then (16) with $C(\psi)$ replaced by $\overline{C}(\psi)$ will yield a mean anomaly degree variance $\overline{c}_\ell$, which is related to $c_\ell$ through the $\beta_\ell$ equations:

$$\overline{c}_\ell = \beta_\ell^2 c_\ell \tag{17}$$

Thus, knowing $\overline{C}(\psi)$ we can find $\overline{c}_\ell$ from (16) and $c_\ell$ from (17) knowing the size of the anomaly blocks to which $\overline{C}(\psi)$ refers. Specifically we can write:

$$c_\ell = \frac{2\ell+1}{2} \frac{1}{\beta_\ell^2 s^{(\ell+2)}} \int_0^\pi \overline{C}(\psi) P_\ell(\cos\psi) \sin\psi \, d\psi \tag{16A}$$

## 3. Numerical 1° Covariance Functions

We first start our numerical determinations by the estimation of the covariance function for 1° (approximately) equal area anomalies. One degree covariance functions have been previously estimated for $\psi$ values from 0° to 7° by Kaula (1966c) and by

Rapp (1972). The values found in the past studies were based on analyzing $1^\circ$ anomalies within a $5^\circ$ equal area anomaly so that product pairs in adjacent $5^\circ$ blocks were not computed nor were product pairs for distances greater than $\psi$ approximately $7^\circ$ were considered. In addition, a programming error made the results of Kaula and Rapp somewhat erroneous.

Because of the limitations of previous estimations of the $1^\circ$ covariance function it was decided that it was appropriate to compute a global $1^\circ$ covariance function. The starting point was a recent collection of 29960, $1^\circ \times 1^\circ$ equiangular mean free-air anomalies that was obtained by updating a $1^\circ \times 1^\circ$ mean anomaly set supplied by the Defense Mapping Agency - Aerospace Center. The updating was carried out using additional data along the lines of a previous update as described in Rapp (1972). These anomalies were all referred to the gravity formula of the Geodetic Reference System 1967. The $1^\circ \times 1^\circ$ equiangular tape was then converted to a set of 21828 (approximately) equal area anomalies. The subdivisions of these anomalies was such that the latitude increment was $1^\circ$ while the longitude increment was some integer degree of such size that the block was approximately equal in area to a $1^\circ \times 1^\circ$ block at the equator. The covariances were computed using equation (3) with the $\Delta\psi$ in equation (5) of $1^\circ$. The results of this computation are given in Table A of the appendix. In this table the following quantities are given: number of product pairs, average $\psi$ (in degrees), covariance ($mgal^2$). For further use the 181 values given in Table A were interpolated to determine a covariance at 0.5 degree intervals. This interpolation was carried out using an Aitken-Lagrange interpolation using 20 points as implemented through subroutine DALI (and DATSG) of the IBM System/360 Scientific Subroutine Package (H20-0205-3), Version III. The resultant 361 values are given in Table One, being identified as the unmodified $\overline{C}(\psi)$ values. The plot of this covariance function is shown in Figure One.

From these unmodified $\overline{C}(\psi)$ values we can compute the smoothed anomaly degree variances from equation (16). Such values are shown for degree 0 through 10 in Table Two where s and $\beta$ are taken to be one (causing a maximum error of less than 5%). In addition values of $c_\ell$ from the recommended set of potential coefficients given by Rapp (1973b) are given for comparison purposes.

Figure 1

1° Covariance Function

A) As originally derived

B) As modified to reflect better anomaly degree variances at degrees 1→10

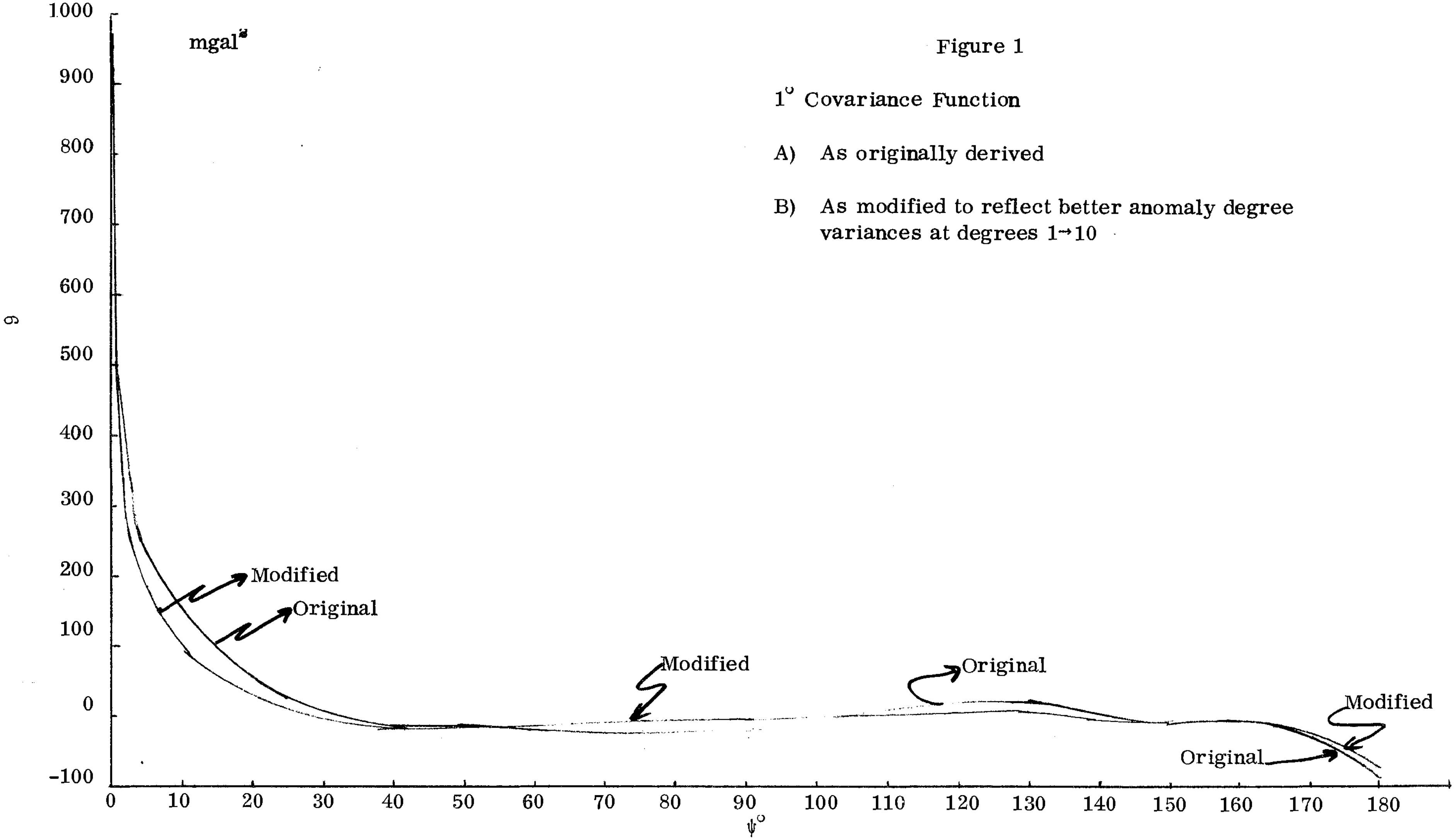


Table One

One Degree Anomaly Covariance Function

| $\psi^\circ$ | $\overline{C}(\psi)$ Modified | $\overline{C}(\psi)$ Unmodified | $\psi^\circ$ | $\overline{C}(\psi)$ Modified | $\overline{C}(\psi)$ Unmodified |
|---|---|---|---|---|---|
| 0.0 | 919.66 | 996.66 | 25.00 | 10.51 | 29.16 |
| 0.50 | 671.64 | 748.60 | 25.50 | 8.90 | 26.48 |
| 1.00 | 493.43 | 570.27 | 26.00 | 6.63 | 23.19 |
| 1.50 | 368.24 | 444.87 | 26.50 | 4.67 | 20.26 |
| 2.00 | 285.35 | 361.70 | 27.00 | 2.99 | 17.65 |
| 2.50 | 236.09 | 312.07 | 27.50 | 1.65 | 15.42 |
| 3.00 | 211.42 | 286.95 | 28.00 | 0.47 | 13.40 |
| 3.50 | 200.69 | 275.70 | 28.50 | -0.86 | 11.28 |
| 4.00 | 193.37 | 267.78 | 29.00 | -2.23 | 9.16 |
| 4.50 | 176.88 | 250.62 | 29.50 | -2.86 | 7.83 |
| 5.00 | 155.86 | 228.85 | 30.00 | -3.29 | 6.73 |
| 5.50 | 146.38 | 218.55 | 30.50 | -4.57 | 4.84 |
| 6.00 | 141.39 | 212.67 | 31.00 | -5.84 | 2.99 |
| 6.50 | 133.39 | 203.72 | 31.50 | -6.17 | 2.12 |
| 7.00 | 124.92 | 194.23 | 32.00 | -6.14 | 1.65 |
| 7.50 | 119.86 | 188.09 | 32.50 | -6.46 | 0.87 |
| 8.00 | 117.38 | 184.48 | 33.00 | -7.38 | -0.48 |
| 8.50 | 116.89 | 182.80 | 33.50 | -8.88 | -2.37 |
| 9.00 | 115.10 | 179.77 | 34.00 | -10.61 | -4.46 |
| 9.50 | 107.02 | 170.40 | 34.50 | -11.85 | -6.03 |
| 10.00 | 96.52 | 158.56 | 35.00 | -12.71 | -7.19 |
| 10.50 | 92.45 | 153.11 | 35.50 | -13.52 | -8.28 |
| 11.00 | 90.54 | 149.79 | 36.00 | -14.16 | -9.17 |
| 11.50 | 82.86 | 140.66 | 36.50 | -14.36 | -9.60 |
| 12.00 | 74.57 | 130.89 | 37.00 | -14.49 | -9.93 |
| 12.50 | 72.40 | 127.21 | 37.50 | -15.11 | -10.74 |
| 13.00 | 71.77 | 125.06 | 38.00 | -15.73 | -11.53 |
| 13.50 | 66.85 | 118.59 | 38.50 | -15.39 | -11.35 |
| 14.00 | 59.75 | 109.92 | 39.00 | -14.70 | -10.81 |
| 14.50 | 53.61 | 102.21 | 39.50 | -14.94 | -11.18 |
| 15.00 | 49.30 | 96.32 | 40.00 | -15.41 | -11.78 |
| 15.50 | 47.01 | 92.44 | 40.50 | -14.59 | -11.08 |
| 16.00 | 45.95 | 89.79 | 41.00 | -13.60 | -10.20 |
| 16.50 | 44.04 | 86.29 | 41.50 | -13.46 | -10.18 |
| 17.00 | 41.61 | 82.29 | 42.00 | -13.36 | -10.19 |
| 17.50 | 39.27 | 78.38 | 42.50 | -12.28 | -9.22 |
| 18.00 | 36.96 | 74.51 | 43.00 | -11.10 | -8.15 |
| 18.50 | 34.87 | 70.88 | 43.50 | -10.96 | -8.13 |
| 19.00 | 32.97 | 67.46 | 44.00 | -11.32 | -8.61 |
| 19.50 | 31.09 | 64.08 | 44.50 | -11.53 | -8.95 |
| 20.00 | 29.33 | 60.85 | 45.00 | -11.94 | -9.49 |
| 20.50 | 27.89 | 57.96 | 45.50 | -12.80 | -10.49 |
| 21.00 | 26.35 | 55.01 | 46.00 | -13.75 | -11.60 |
| 21.50 | 24.11 | 51.38 | 46.50 | -14.14 | -12.15 |
| 22.00 | 21.59 | 47.51 | 47.00 | -14.26 | -12.44 |
| 22.50 | 19.81 | 44.42 | 47.50 | -14.75 | -13.12 |
| 23.00 | 18.06 | 41.40 | 48.00 | -15.25 | -13.81 |
| 23.50 | 14.99 | 37.09 | 48.50 | -15.06 | -13.83 |
| 24.00 | 11.78 | 32.69 | 49.00 | -14.93 | -13.92 |
| 24.50 | 10.79 | 30.54 | 49.50 | -16.34 | -15.56 |

| | | |
|---|---|---|
| 50.00 | -17.93 | -17.39 |
| 50.50 | -18.22 | -17.94 |
| 51.00 | -18.07 | -18.06 |
| 51.50 | -18.45 | -18.72 |
| 52.00 | -19.03 | -19.59 |
| 52.50 | -19.34 | -20.21 |
| 53.00 | -19.19 | -20.37 |
| 53.50 | -18.43 | -19.94 |
| 54.00 | -17.53 | -19.37 |
| 54.50 | -17.29 | -19.47 |
| 55.00 | -17.42 | -19.95 |
| 55.50 | -17.28 | -20.17 |
| 56.00 | -17.00 | -20.25 |
| 56.50 | -16.64 | -20.26 |
| 57.00 | -16.03 | -20.02 |
| 57.50 | -14.87 | -19.23 |
| 58.00 | -13.61 | -18.35 |
| 58.50 | -12.86 | -17.98 |
| 59.00 | -12.44 | -17.93 |
| 59.50 | -12.37 | -18.24 |
| 60.00 | -12.46 | -18.71 |
| 60.50 | -12.51 | -19.13 |
| 61.00 | -12.38 | -19.37 |
| 61.50 | -11.79 | -19.14 |
| 62.00 | -10.88 | -18.59 |
| 62.50 | -9.92 | -17.99 |
| 63.00 | -9.23 | -17.64 |
| 63.50 | -9.17 | -17.92 |
| 64.00 | -9.42 | -18.51 |
| 64.50 | -9.34 | -18.75 |
| 65.00 | -9.14 | -18.87 |
| 65.50 | -9.21 | -19.24 |
| 66.00 | -9.30 | -19.63 |
| 66.50 | -8.92 | -19.53 |
| 67.00 | -8.50 | -19.39 |
| 67.50 | -8.50 | -19.65 |
| 68.00 | -8.82 | -20.23 |
| 68.50 | -9.26 | -20.91 |
| 69.00 | -9.53 | -21.42 |
| 69.50 | -9.29 | -21.40 |
| 70.00 | -8.84 | -21.16 |
| 70.50 | -8.71 | -21.24 |
| 71.00 | -8.73 | -21.45 |
| 71.50 | -9.00 | -21.90 |
| 72.00 | -9.12 | -22.20 |
| 72.50 | -8.62 | -21.86 |
| 73.00 | -7.71 | -21.10 |
| 73.50 | -6.77 | -20.31 |
| 74.00 | -6.07 | -19.75 |
| 74.50 | -5.86 | -19.67 |
| 75.00 | -6.12 | -20.06 |
| 75.50 | -6.72 | -20.77 |
| 76.00 | -7.34 | -21.50 |
| 76.50 | -7.60 | -21.87 |
| 77.00 | -7.61 | -21.97 |
| 77.50 | -7.66 | -22.12 |
| 78.00 | -7.55 | -22.09 |
| 78.50 | -6.83 | -21.45 |
| 79.00 | -5.97 | -20.67 |
| 79.50 | -5.87 | -20.64 |
| 80.00 | -5.91 | -20.75 |
| 80.50 | -5.93 | -20.83 |
| 81.00 | -6.02 | -20.98 |
| 81.50 | -6.71 | -21.72 |
| 82.00 | -7.14 | -22.20 |
| 82.50 | -6.80 | -21.90 |
| 83.00 | -6.47 | -21.60 |
| 83.50 | -6.15 | -21.31 |
| 84.00 | -5.83 | -21.02 |
| 84.50 | -6.01 | -21.22 |
| 85.00 | -6.04 | -21.26 |
| 85.50 | -5.17 | -20.39 |
| 86.00 | -4.16 | -19.37 |
| 86.50 | -3.54 | -18.74 |
| 87.00 | -3.09 | -18.26 |
| 87.50 | -2.75 | -17.89 |
| 88.00 | -2.55 | -17.64 |
| 88.50 | -2.57 | -17.60 |
| 89.00 | -2.51 | -17.47 |
| 89.50 | -2.12 | -17.00 |
| 90.00 | -1.78 | -16.56 |
| 90.50 | -1.80 | -16.46 |
| 91.00 | -1.70 | -16.23 |
| 91.50 | -1.10 | -15.48 |
| 92.00 | -0.52 | -14.73 |
| 92.50 | -0.47 | -14.49 |
| 93.00 | -0.25 | -14.07 |
| 93.50 | 0.45 | -13.14 |
| 94.00 | 1.03 | -12.31 |
| 94.50 | 1.19 | -11.88 |
| 95.00 | 1.47 | -11.31 |
| 95.50 | 2.13 | -10.34 |
| 96.00 | 2.83 | -9.30 |
| 96.50 | 3.36 | -8.41 |
| 97.00 | 3.82 | -7.56 |
| 97.50 | 4.24 | -6.73 |
| 98.00 | 4.83 | -5.71 |
| 98.50 | 5.65 | -4.43 |
| 99.00 | 6.56 | -3.03 |
| 99.50 | 7.42 | -1.67 |
| 100.00 | 7.97 | -0.59 |
| 100.50 | 8.07 | 0.06 |
| 101.00 | 8.42 | 0.99 |
| 101.50 | 9.67 | 2.83 |
| 102.00 | 10.61 | 4.39 |
| 102.50 | 10.21 | 4.62 |
| 103.00 | 9.79 | 4.85 |
| 103.50 | 10.04 | 5.77 |
| 104.00 | 10.26 | 6.68 |
| 104.50 | 10.02 | 7.14 |

| | | |
|---|---|---|
| 105.00 | 9.38 | 7.21 |
| 105.50 | 8.38 | 6.93 |
| 106.00 | 7.64 | 6.92 |
| 106.50 | 7.71 | 7.73 |
| 107.00 | 7.90 | 8.66 |
| 107.50 | 7.82 | 9.33 |
| 108.00 | 7.85 | 10.11 |
| 108.50 | 8.19 | 11.19 |
| 109.00 | 8.60 | 12.34 |
| 109.50 | 8.92 | 13.40 |
| 110.00 | 9.20 | 14.41 |
| 110.50 | 9.28 | 15.21 |
| 111.00 | 9.61 | 16.24 |
| 111.50 | 10.34 | 17.67 |
| 112.00 | 10.95 | 18.95 |
| 112.50 | 11.10 | 19.76 |
| 113.00 | 10.80 | 20.09 |
| 113.50 | 10.20 | 20.10 |
| 114.00 | 9.87 | 20.36 |
| 114.50 | 10.31 | 21.36 |
| 115.00 | 10.51 | 22.09 |
| 115.50 | 9.69 | 21.76 |
| 116.00 | 8.95 | 21.49 |
| 116.50 | 8.85 | 21.82 |
| 117.00 | 8.69 | 22.06 |
| 117.50 | 8.01 | 21.74 |
| 118.00 | 7.38 | 21.43 |
| 118.50 | 6.63 | 20.96 |
| 119.00 | 6.24 | 20.82 |
| 119.50 | 6.58 | 21.36 |
| 120.00 | 6.96 | 21.90 |
| 120.50 | 7.53 | 22.59 |
| 121.00 | 7.87 | 23.02 |
| 121.50 | 7.80 | 22.99 |
| 122.00 | 7.26 | 22.45 |
| 122.50 | 6.29 | 21.44 |
| 123.00 | 5.32 | 20.38 |
| 123.50 | 4.62 | 19.57 |
| 124.00 | 4.55 | 19.34 |
| 124.50 | 5.22 | 19.81 |
| 125.00 | 5.63 | 20.00 |
| 125.50 | 5.11 | 19.21 |
| 126.00 | 4.67 | 18.47 |
| 126.50 | 4.86 | 18.34 |
| 127.00 | 5.34 | 18.46 |
| 127.50 | 5.89 | 18.63 |
| 128.00 | 6.64 | 18.97 |
| 128.50 | 7.62 | 19.52 |
| 129.00 | 8.77 | 20.21 |
| 129.50 | 10.03 | 21.00 |
| 130.00 | 10.83 | 21.32 |
| 130.50 | 10.78 | 20.76 |
| 131.00 | 10.11 | 19.58 |
| 131.50 | 9.12 | 18.07 |
| 132.00 | 8.14 | 16.56 |

| | | |
|---|---|---|
| 132.50 | 7.45 | 15.34 |
| 133.00 | 7.40 | 14.76 |
| 133.50 | 8.11 | 14.93 |
| 134.00 | 8.88 | 15.17 |
| 134.50 | 8.98 | 14.75 |
| 135.00 | 8.83 | 14.08 |
| 135.50 | 8.89 | 13.64 |
| 136.00 | 8.15 | 12.40 |
| 136.50 | 6.01 | 9.78 |
| 137.00 | 4.15 | 7.46 |
| 137.50 | 3.76 | 6.62 |
| 138.00 | 3.65 | 6.08 |
| 138.50 | 3.13 | 5.15 |
| 139.00 | 2.63 | 4.26 |
| 139.50 | 2.36 | 3.63 |
| 140.00 | 1.84 | 2.77 |
| 140.50 | 0.87 | 1.48 |
| 141.00 | 0.18 | 0.50 |
| 141.50 | 0.22 | 0.27 |
| 142.00 | 0.23 | 0.04 |
| 142.50 | 0.28 | -0.13 |
| 143.00 | 0.12 | -0.48 |
| 143.50 | -0.35 | -1.12 |
| 144.00 | -0.87 | -1.78 |
| 144.50 | -1.24 | -2.27 |
| 145.00 | -1.76 | -2.88 |
| 145.50 | -2.56 | -3.75 |
| 146.00 | -3.42 | -4.66 |
| 146.50 | -4.20 | -5.48 |
| 147.00 | -4.92 | -6.21 |
| 147.50 | -5.60 | -6.88 |
| 148.00 | -5.99 | -7.25 |
| 148.50 | -5.94 | -7.17 |
| 149.00 | -6.01 | -7.19 |
| 149.50 | -6.67 | -7.79 |
| 150.00 | -6.72 | -7.77 |
| 150.50 | -5.28 | -6.26 |
| 151.00 | -3.91 | -4.81 |
| 151.50 | -3.45 | -4.27 |
| 152.00 | -3.48 | -4.22 |
| 152.50 | -3.75 | -4.40 |
| 153.00 | -3.97 | -4.54 |
| 153.50 | -4.01 | -4.51 |
| 154.00 | -4.55 | -4.98 |
| 154.50 | -6.07 | -6.44 |
| 155.00 | -6.94 | -7.26 |
| 155.50 | -5.86 | -6.15 |
| 156.00 | -4.85 | -5.11 |
| 156.50 | -4.94 | -5.19 |
| 157.00 | -4.74 | -5.00 |
| 157.50 | -3.69 | -3.97 |
| 158.00 | -3.57 | -3.89 |
| 158.50 | -5.32 | -5.70 |
| 159.00 | -6.65 | -7.11 |
| 159.50 | -6.48 | -7.04 |

| | | |
|---|---|---|
| 160.00 | -6.07 | -6.75 |
| 160.50 | -6.19 | -7.02 |
| 161.00 | -7.31 | -8.30 |
| 161.50 | -9.38 | -10.55 |
| 162.00 | -10.89 | -12.27 |
| 162.50 | -10.66 | -12.26 |
| 163.00 | -10.96 | -12.81 |
| 163.50 | -13.54 | -15.66 |
| 164.00 | -15.49 | -17.89 |
| 164.50 | -15.45 | -18.15 |
| 165.00 | -14.70 | -17.72 |
| 165.50 | -14.05 | -17.40 |
| 166.00 | -14.10 | -17.80 |
| 166.50 | -14.96 | -19.01 |
| 167.00 | -15.07 | -19.49 |
| 167.50 | -13.53 | -18.33 |
| 168.00 | -12.14 | -17.32 |
| 168.50 | -12.07 | -17.64 |
| 169.00 | -12.91 | -18.88 |
| 169.50 | -14.92 | -21.28 |
| 170.00 | -17.18 | -23.93 |
| 170.50 | -19.16 | -26.30 |
| 171.00 | -19.96 | -27.49 |
| 171.50 | -19.48 | -27.39 |
| 172.00 | -19.95 | -28.23 |
| 172.50 | -23.66 | -32.30 |
| 173.00 | -27.41 | -36.39 |
| 173.50 | -29.62 | -38.93 |
| 174.00 | -30.53 | -40.16 |
| 174.50 | -30.66 | -40.59 |
| 175.00 | -30.88 | -41.08 |
| 175.50 | -31.99 | -42.45 |
| 176.00 | -33.69 | -44.38 |
| 176.50 | -36.24 | -47.14 |
| 177.00 | -40.40 | -51.48 |
| 177.50 | -47.02 | -58.26 |
| 178.00 | -54.21 | -65.58 |
| 178.50 | -54.64 | -66.11 |
| 179.00 | -41.72 | -53.26 |
| 179.50 | -37.87 | -49.46 |
| 180.00 | -72.83 | -84.43 |

Table Two

Smoothed Anomaly Degree Variances ($c_\ell$) As
Computed With 1° Free Air Anomalies
and From a Current Potential Coefficient Set

($mgal^2$)

| Degree | $c_\ell$ from 1° anomaly data | $c_\ell$ from potential coefficients (Rapp, 1973b) |
|---|---|---|
| 0 | 0.07 | -- |
| 1 | 2.3 | -- |
| 2 | 26.2 | 7.5 |
| 3 | 58.3 | 33.9 |
| 4 | 16.0 | 19.2 |
| 5 | 26.3 | 21.6 |
| 6 | 36.0 | 18.9 |
| 7 | 22.8 | 18.8 |
| 8 | 12.6 | 10.4 |
| 9 | 20.0 | 11.1 |
| 10 | 9.3 | 11.4 |

If the gravity formula were that of a mean earth ellipsoid, the zeroth degree variance should be zero. This is essentially the case here with the fact that the $\gamma_e$ and the flattening of the GRS67 are quite close to be current best estimates of these parameters (Rapp, 1974). The anomalies taken on a global scale should have no first degree anomaly degree variance. The non-global 1° anomalies that we have imply through the covariance function a small one of 2.3 $mgal^2$.

The anomaly degree variances from the potential coefficients should be reliable at the lower degrees because of the accurate determination of low degree potential coefficients through satellite orbital analysis. Comparison of these values with that implied by the covariance function indicates poor agreement for degrees 2, 3, 6 and 9. This disagreement may be related to the fact that the 1° anomalies cover only 50% of the earth's surface and we cannot hope to find good low degree information from such limited coverage.

However, for future analysis it is important that we use a 1° covariance function that is characteristic of the real world especially at low degrees. To develop such a covariance function we modify the covariance function computed from the anomalies by imposing on the modified function the $c_\ell$ values to degree 10 as listed in Table Two (as computed from potential coefficients). To do this we first remove the effect of the $\overline{c}_\ell$ values listed in Table Two and then add back the covariance contribution from the $c_\ell$ values, in both cases using equation (11) setting $\beta_\ell$ and s equal to one. In effect we carry out the following modification to obtain a modified 1° covariance function:

$$\overline{C(\psi)}_m = \overline{C(\psi)}_{ORIG} + \sum_{\ell=1}^{10} \left(c_{\ell_{(POT\ COEFF)}} - c_{\ell_{(ORIG)}}\right) P_\ell(\cos\psi) \tag{18}$$

The modified covariance function is shown in Table One being labeled Modified $\overline{C}(\psi)$. This modified covariance function is plotted in Figure One.

Smoothed anomaly degree variances were developed from this modified covariance function where were then converted to the actual degree variances using equation (16A). These results and values of $\beta_\ell$ for one degree blocks and $s^{-(\ell+2)}$ are given in Table B of the appendix.

From Table One, using the modified covariance function of the current estimate for the variance of a 1° anomaly is 919.66 mgal², or a root mean square value of ±30.3 mgals with respect to the gravity formula of the Geodetic Reference System 1967.

## 4. A Five Degree Anomaly Variance

For purposes of obtaining models of anomaly degree variance using procedures such as described in Rapp (1973a) we need to estimate the variance of the 5° anomalies. This can be done in two ways. The first procedure is by the numerical integration of the 1° modified covariance function according to equation (7-82) of Heiskanen and Moritz (p. 270). This leads to an estimate of 305 mgal². The second procedure is to compute the variance directly from the 5° anomalies. This was done by first predicting 5° equal area anomalies using the methods described in Rapp (1972) but with the more current 1°x 1° set. The variance computed by this procedure from the 1354 predicted anomalies was 298 mgal². We adopt for further use the variance of 5 degree anomalies as 302 mgal² with respect to the gravity formula of the Geodetic Reference System 1967.

## 5. The Point Anomaly Variance

The value of $C_0$ is an important quantity as it is a scaling factor for many representations of the point anomaly covariance function. $C_0$ has been treated as both a local or regional quantity, or a global quantity. On a regional basis $C_0$ is the variance of the point anomalies in some defined area. Thus, it will change from area to area. The global $C_0$ value is considered to be representative of the gravity field of the whole earth. The estimation of $C_0$ on a global basis is not straight forward since we do not have global gravity coverage. The only global point covariance function numerically estimated is that given by Kaula (1959) where he used gravity data that was current to 1958. During the 16 years since the compliation of gravity data as used by Kaula, a considerable amount of additional data has become available. Thus, a new computation of global point covariance seems appropriate and is needed. Such a computation can only be done through some organization that has access to the gravity data holdings. For this report we do not have the facilities or funds to carry out a computation of a point covariance function. However, we can use several procedures to determine $C_0$, the quantity so fundamental to the analytical representation of a point covariance function.

## 5.1 Method One

One method to estimate $C_0$ is to consider the relationship between a point covariance function (C(d)) and the variance ($G^2_{s^\circ}$) of anomalies given in blocks of size $s^\circ$. One convenient relationship is given by Hirvonen (1962):

$$G^2_{s^\circ} = \int_0^{\sqrt{2}} W\,C\,(d)\,dr \tag{19}$$

where $d = rs^\circ$

$W = (2\pi - 8r + 2r^2)r$ when $0 < r < 1$

$W = (2\pi - 4 - 2r^2 + 8\sqrt{r^2 - 1} \;\; -8\tan^{-1}\sqrt{r^2 - 1})\;r$ where $1 < r < \sqrt{2}$

If we represent C(d) in the form of:

$$C(d) = C_0\,f(d) \tag{20}$$

we can solve (19) and (20) for $C_0$:

$$C_0 = \frac{G^2_{s^\circ}}{\int_0^{\sqrt{2}} Wf(d)\,dr} \equiv \frac{G^2_{s^\circ}}{I} \tag{21}$$

The value of I can be obtained for various representative f(d).

Many representations of the point covariance function have been suggested. Many of these representations are summarized in papers by Groten (1966), Lauer (1971), and Jordan (1972). For the purposes of this paper we have used three models. These are:

$$(1)\quad C(d) = C_0 \underbrace{e^{-\alpha d}}_{f_1(d)} \tag{22}$$

$$(2)\quad C(d) = \underbrace{C_0 \left(1 - \frac{d}{2c_2}\right) e^{-d/c_2}}_{f_2(d)} \tag{23}$$

$$(3)\quad C(d) = C_0\,\underbrace{(1+d(a_1 + d(a_2 + d(a_3 + d(a_4 + d(a_5))))))}_{f_3(d)} \tag{24}$$

The $c_1$ and $c_2$ values were obtained from fitting the Kaula (1959) point covariance curve to a distance of 1.5°. We found $c_1 = 0^\circ.897$ and $c_2 = 1^\circ.88$. Beyond a distance of 1.5°, the point covariance would not be represented well by equations (22) and (23) with the above constants. The constants of equation (24) were obtained by a least squares polynomial fit using the Kaula point covariance function to 8°. We found:

$a_1 = -.9816195$
$a_2 = .4894498$
$a_3 = -.1149583$
$a_4 = .0126057$
$a_5 = -.000523222$

For these models, the root mean square fit to the observed covariance function was $\pm 30$ mgal², $\pm 75$ mgal², and $\pm 28$ mgal² for models 1, 2 and 3 respectively. For $s^\circ = 1^\circ$, values of I (computed by numerical integration), and $C_0$(taking $G_s^2 = 919.66$ from Table One) are given for each of the models in Table Three.

Table Three
Estimation of $C_0$ from 1°
Anomaly Block Variances

| Model | I | $C_0$ |
|---|---|---|
| 1, equation (22) | .64185 | 1433 mgal² |
| 2, equation (23) | .66491 | 1383 mgal² |
| 3, equation (24) | .62595 | 1469 mgal² |

Using weights based on the root mean square fits to the point covariance curve, the estimated $C_0$ from this analysis is 1447 mgal².

## 5.2 Method Two

A more direct method for determining $C_0$ is through the analysis of the actual point gravity anomalies. Such an analysis is not a straight forward one since the anomaly data is not uniformly distributed over the earth. Since certain areas (such as land areas) have, in general, denser anomaly coverage than ocean areas, and since free-air anomalies are correlated with land elevations or ocean depth, special care needs to be taken in the analysis of a set of point gravity anomalies for $C_0$.

In our analysis we basically considered a point variance by elevation range, and then converted these individual variances into a global estimate of $C_0$ by forming a weighted mean with weights being based on the percentage of the earth's surface lying within the elevation range.

As the first step in this procedure the Defense Mapping Agency Aerospace Center considered a set of 2,253,122 point free-air anomalies whose elevation or depth was known. Elevation ranges of 100 meter increment were chosen. For all anomalies falling within each range, the mean anomaly, the mean square anomaly and the mean elevation from the points, was determined. The mean square anomaly was computed as the sum of the square of the anomalies with the elevation range divided by the number of anomalies within the range. In subsequent discussions this quantity will be referred to as the variance of the range. This terminology is not specifically correct as a variance is usually defined with respect to a quantity whose mean is zero. In fact, the anomaly mean within a range will not be zero, but it will be zero or close to it on a global basis. This data by ranges is shown in Table Four.

In order to form a global estimate of $C_0$, we now need to know how elevations are distributed on the actual earth. To do this we considered mean elevations in 1654 $5^\circ$ equal area blocks and 64800, $1^\circ \times 1^\circ$ mean elevations. From this data the percentage of the earth's surface within a given elevation range could be found. The results found for the $5^\circ$ and $1^\circ$ data are shown as the last two columns in Table Four. The $5^\circ$ results are shown as a matter of interest only, as the $5^\circ$ subdivision is too large for the purposes needed here. We should note that all 0.0's given in Table Four with the exception of the mean anomaly for the 100 to 200 meter range indicate no data was available for the quantity. The $1^\circ$ subdivision is also not sufficiently small for the most accurate work as can be seen from the fact that certain elevation ranges for which there was point elevations data were not represented in the data from the $1^\circ$ mean elevation data.

The weighted variance (or $C_0$) was then determined as follows:

$$C_0 = \frac{\sum_i P_i (C_0)_i}{\sum_i P_i} \tag{25}$$

Table Four
Anomaly Variance and Related Information by Elevation Range

| Elevation Range (meters) | | No. of point Anomalies | Point anom. mean (mgals) | Mean sq. anomaly ($mgal^2$) | Average of pt. elevations (meters) | Percentage of earth's surface within range $1^\circ$ data | $5^\circ$ data |
|---|---|---|---|---|---|---|---|
| -14100 | -14000 | 0 | 0.0 | 0.0 | 0.0 | 0.002 | 0.0 |
| -11200 | -11100 | 1 | -213.1 | 45411.6 | -11113.0 | 0.0 | 0.0 |
| -10800 | -10700 | 5 | -300.8 | 90650.3 | -10750.6 | 0.0 | 0.0 |
| -10700 | -10600 | 1 | -277.3 | 76895.3 | -10674.0 | 0.0 | 0.0 |
| -10600 | -10500 | 2 | -285.0 | 81253.5 | -10592.0 | 0.0 | 0.0 |
| -10500 | -10400 | 5 | -282.4 | 79833.0 | -10425.2 | 0.0 | 0.0 |
| -10400 | -10300 | 13 | -270.6 | 74429.2 | -10353.8 | 0.0 | 0.0 |
| -10300 | -10200 | 8 | -290.3 | 85035.7 | -10228.4 | 0.0 | 0.0 |
| -10200 | -10100 | 12 | -283.4 | 80914.0 | -10149.4 | 0.0 | 0.0 |
| -10100 | -10000 | 9 | -282.3 | 80402.3 | -10065.4 | 0.0 | 0.0 |
| -10000 | -9900 | 22 | -279.8 | 79255.9 | -9947.7 | 0.0 | 0.0 |
| -9900 | -9800 | 7 | -276.4 | 76443.4 | -9858.6 | 0.0 | 0.0 |
| -9800 | -9700 | 11 | -273.5 | 77105.5 | -9743.8 | 0.0 | 0.0 |
| -9700 | -9600 | 19 | -260.5 | 70629.0 | -9645.9 | 0.0 | 0.0 |
| -9600 | -9500 | 16 | -267.7 | 73083.8 | -9546.7 | 0.0 | 0.0 |
| -9500 | -9400 | 21 | -248.8 | 64161.0 | -9446.1 | 0.0 | 0.0 |
| -9400 | -9300 | 27 | -241.9 | 63265.9 | -9347.8 | 0.0 | 0.0 |
| -9300 | -9200 | 17 | -259.2 | 68675.0 | -9247.8 | 0.0 | 0.0 |
| -9200 | -9100 | 22 | -231.9 | 56997.3 | -9147.5 | 0.0 | 0.0 |
| -9100 | -9000 | 24 | -243.2 | 61142.7 | -9041.5 | 0.0 | 0.0 |
| -9000 | -8900 | 30 | -236.8 | 57878.9 | -8956.7 | 0.0 | 0.0 |
| -8900 | -8800 | 24 | -242.4 | 61454.6 | -8853.9 | 0.0 | 0.0 |
| -8800 | -8700 | 36 | -222.6 | 52069.2 | -8764.4 | 0.0 | 0.0 |
| -8700 | -8600 | 28 | -227.8 | 54300.5 | -8667.9 | 0.0 | 0.0 |
| -8600 | -8500 | 43 | -222.1 | 51648.5 | -8551.8 | 0.0 | 0.0 |
| -8500 | -8400 | 56 | -225.5 | 52099.5 | -8440.7 | 0.0 | 0.0 |
| -8400 | -8300 | 386 | -249.9 | 63345.2 | -8360.1 | 0.0 | 0.0 |
| -8300 | -8200 | 130 | -220.6 | 50153.0 | -8259.2 | 0.0 | 0.0 |
| -8200 | -8100 | 156 | -214.7 | 47255.5 | -8148.5 | 0.0 | 0.0 |
| -8100 | -8000 | 239 | -221.0 | 51087.8 | -8048.0 | 0.0 | 0.0 |
| -8000 | -7900 | 277 | -231.6 | 57988.7 | -7952.1 | 0.0 | 0.0 |
| -7900 | -7800 | 310 | -211.2 | 47870.1 | -7856.2 | 0.002 | 0.0 |
| -7800 | -7700 | 220 | -207.7 | 46967.6 | -7759.5 | 0.0 | 0.0 |
| -7700 | -7600 | 210 | -204.2 | 45255.4 | -7654.5 | 0.002 | 0.0 |
| -7600 | -7500 | 265 | -205.3 | 46034.8 | -7551.2 | 0.002 | 0.0 |
| -7500 | -7400 | 313 | -214.8 | 51871.4 | -7446.8 | 0.0 | 0.0 |
| -7400 | -7300 | 417 | -180.2 | 35771.0 | -7337.9 | 0.008 | 0.0 |
| -7300 | -7200 | 478 | -168.6 | 31597.3 | -7250.2 | 0.003 | 0.0 |
| -7200 | -7100 | 506 | -158.3 | 28832.5 | -7151.5 | 0.002 | 0.0 |
| -7100 | -7000 | 359 | -180.4 | 38391.0 | -7054.5 | 0.014 | 0.0 |
| -7000 | -6900 | 407 | -154.1 | 27775.9 | -6949.7 | 0.007 | 0.0 |
| -6900 | -6800 | 358 | -160.3 | 30243.8 | -6851.3 | 0.002 | 0.0 |
| -6800 | -6700 | 473 | -131.8 | 21597.5 | -6744.5 | 0.005 | 0.0 |
| -6700 | -6600 | 465 | -119.5 | 18588.8 | -6658.1 | 0.009 | 0.0 |
| -6600 | -6500 | 530 | -119.5 | 18717.8 | -6546.2 | 0.024 | 0.0 |
| -6500 | -6400 | 706 | -105.5 | 15090.3 | -6454.0 | 0.010 | 0.0 |
| -6400 | -6300 | 796 | -93.0 | 12701.3 | -6349.6 | 0.027 | 0.0 |
| -6300 | -6200 | 978 | -77.2 | 9847.8 | -6246.4 | 0.054 | 0.0 |
| -6200 | -6100 | 1311 | -57.7 | 6841.8 | -6146.7 | 0.153 | 0.0 |
| -6100 | -6000 | 2596 | -32.4 | 4002.0 | -6049.2 | 0.321 | 0.0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| -6000 | -5900 | 4024 | -27.3 | 3215.6 | -5946.8 | 0.292 | 0.120 |
| -5900 | -5800 | 7166 | -22.5 | 1786.2 | -5847.2 | 0.502 | 0.181 |
| -5800 | -5700 | 8170 | -17.8 | 1319.7 | -5750.8 | 0.697 | 0.403 |
| -5700 | -5600 | 9542 | -14.2 | 1287.4 | -5652.4 | 0.983 | 0.562 |
| -5600 | -5500 | 10061 | -13.0 | 1143.5 | -5550.1 | 1.449 | 0.833 |
| -5500 | -5400 | 12429 | -11.7 | 1106.4 | -5450.9 | 1.170 | 1.059 |
| -5400 | -5300 | 14512 | -12.0 | 1113.8 | -5350.6 | 1.478 | 1.688 |
| -5300 | -5200 | 14791 | -12.2 | 1245.1 | -5248.2 | 2.130 | 1.620 |
| -5200 | -5100 | 15930 | -13.6 | 1243.7 | -5147.5 | 3.033 | 2.111 |
| -5100 | -5000 | 16687 | -12.7 | 1231.6 | -5051.1 | 2.766 | 1.738 |
| -5000 | -4900 | 16185 | -11.9 | 1370.6 | -4950.5 | 2.222 | 2.816 |
| -4900 | -4800 | 16178 | -14.6 | 1401.8 | -4849.1 | 2.395 | 2.085 |
| -4800 | -4700 | 16412 | -13.9 | 1350.3 | -4750.2 | 1.815 | 2.459 |
| -4700 | -4600 | 16153 | -13.2 | 1433.0 | -4651.0 | 2.027 | 2.300 |
| -4600 | -4500 | 16000 | -12.2 | 1243.0 | -4551.8 | 2.115 | 1.613 |
| -4500 | -4400 | 14261 | -12.3 | 1431.1 | -4452.9 | 2.102 | 2.059 |
| -4400 | -4300 | 12350 | -10.6 | 1550.4 | -4352.9 | 2.013 | 2.866 |
| -4300 | -4200 | 12568 | -12.1 | 1588.8 | -4251.1 | 2.578 | 2.051 |
| -4200 | -4100 | 11377 | -15.1 | 2402.4 | -4149.5 | 2.366 | 2.514 |
| -4100 | -4000 | 11111 | -11.0 | 1709.1 | -4050.4 | 2.435 | 2.053 |
| -4000 | -3900 | 11122 | -8.7 | 1476.6 | -3949.8 | 1.649 | 2.129 |
| -3900 | -3800 | 10887 | -10.3 | 1705.2 | -3851.0 | 1.930 | 2.204 |
| -3800 | -3700 | 9948 | -7.6 | 1732.4 | -3751.4 | 1.626 | 1.929 |
| -3700 | -3600 | 9982 | -7.4 | 1861.9 | -3650.7 | 1.404 | 2.398 |
| -3600 | -3500 | 10272 | -7.9 | 2086.7 | -3550.7 | 1.647 | 1.299 |
| -3500 | -3400 | 9574 | -6.8 | 2061.3 | -3451.0 | 1.282 | 2.092 |
| -3400 | -3300 | 9899 | -7.1 | 1974.4 | -3350.6 | 1.220 | 1.449 |
| -3300 | -3200 | 11356 | -5.0 | 1920.2 | -3250.4 | 1.339 | 1.818 |
| -3200 | -3100 | 11342 | -5.0 | 1884.6 | -3151.1 | 1.219 | 1.167 |
| -3100 | -3000 | 11490 | -7.6 | 2039.3 | -3050.1 | 1.673 | 1.091 |
| -3000 | -2900 | 12194 | -10.2 | 2075.9 | -2950.7 | 0.829 | 0.914 |
| -2900 | -2800 | 13157 | -10.6 | 2138.0 | -2848.7 | 0.765 | 0.967 |
| -2800 | -2700 | 13830 | -9.4 | 2080.9 | -2748.9 | 0.874 | 0.714 |
| -2700 | -2600 | 13583 | -12.9 | 2353.4 | -2651.8 | 0.453 | 0.837 |
| -2600 | -2500 | 19490 | -2.2 | 2107.7 | -2544.1 | 0.570 | 0.558 |
| -2500 | -2400 | 10858 | -17.7 | 3254.9 | -2451.0 | 0.909 | 1.047 |
| -2400 | -2300 | 10449 | -15.4 | 3024.0 | -2351.8 | 0.421 | 0.789 |
| -2300 | -2200 | 9355 | -15.0 | 3135.1 | -2251.8 | 0.439 | 1.332 |
| -2200 | -2100 | 12206 | -9.5 | 2133.6 | -2147.7 | 1.018 | 0.593 |
| -2100 | -2000 | 12676 | -2.8 | 1803.5 | -2048.7 | 0.559 | 0.615 |
| -2000 | -1900 | 12963 | 1.2 | 1713.2 | -1951.6 | 0.288 | 0.538 |
| -1900 | -1800 | 11371 | -0.3 | 2125.1 | -1851.6 | 0.682 | 0.322 |
| -1800 | -1700 | 9321 | 0.0 | 2522.1 | -1751.1 | 0.330 | 0.659 |
| -1700 | -1600 | 9475 | 5.2 | 2308.1 | -1649.8 | 0.302 | 0.488 |
| -1600 | -1500 | 10452 | 6.1 | 1964.9 | -1549.9 | 0.563 | 0.790 |
| -1500 | -1400 | 10102 | 4.9 | 2023.2 | -1453.0 | 0.241 | 0.673 |
| -1400 | -1300 | 10371 | 5.4 | 2321.7 | -1350.6 | 0.252 | 0.547 |
| -1300 | -1200 | 10581 | 3.8 | 2407.0 | -1252.6 | 0.504 | 0.721 |
| -1200 | -1100 | 9917 | 6.1 | 2397.7 | -1151.8 | 0.251 | 0.431 |
| -1100 | -1000 | 9722 | 7.8 | 2246.5 | -1051.8 | 0.439 | 0.892 |
| -1000 | -900 | 9816 | 11.5 | 2221.2 | -949.4 | 0.383 | 0.766 |
| -900 | -800 | 11381 | 9.2 | 2640.5 | -848.5 | 0.331 | 0.720 |
| -800 | -700 | 10731 | 11.2 | 2043.0 | -752.6 | 0.288 | 0.661 |
| -700 | -600 | 9038 | 14.2 | 2294.5 | -650.6 | 0.400 | 0.570 |
| -600 | -500 | 10338 | 16.9 | 2636.4 | -547.9 | 0.459 | 0.604 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| -500 | -400 | 12816 | 15.4 | 2470.5 | -447.7 | 0.394 | 0.779 |
| -400 | -300 | 16341 | 12.9 | 2490.7 | -348.5 | 0.696 | 0.371 |
| -300 | -200 | 19910 | 12.9 | 1938.9 | -249.7 | 0.772 | 0.528 |
| -200 | -100 | 37357 | 13.6 | 1756.5 | -141.3 | 1.073 | 1.091 |
| -100 | 0 | 85482 | 8.7 | 1713.2 | -49.1 | 3.151 | 1.925 |
| 0 | 100 | 404177 | 3.1 | 1345.0 | 40.6 | 3.557 | 2.825 |
| 100 | 200 | 227862 | 0.0 | 807.1 | 147.8 | 3.961 | 3.477 |
| 200 | 300 | 172980 | -0.6 | 801.3 | 245.2 | 3.431 | 3.821 |
| 300 | 400 | 106121 | -0.8 | 970.9 | 347.6 | 2.960 | 2.447 |
| 400 | 500 | 86419 | 0.5 | 1054.6 | 448.2 | 2.424 | 2.855 |
| 500 | 600 | 51225 | 3.3 | 1345.5 | 546.7 | 1.812 | 2.197 |
| 600 | 700 | 35994 | 1.4 | 1580.4 | 647.9 | 1.497 | 0.990 |
| 700 | 800 | 29210 | -2.2 | 1654.3 | 748.6 | 1.210 | 1.272 |
| 800 | 900 | 26750 | 1.8 | 1540.9 | 849.5 | 1.109 | 0.717 |
| 900 | 1000 | 23329 | 2.4 | 1540.5 | 948.9 | 1.054 | 1.277 |
| 1000 | 1100 | 23078 | 5.6 | 1416.9 | 1048.9 | 0.889 | 0.898 |
| 1100 | 1200 | 26348 | -0.6 | 1193.7 | 1154.8 | 0.773 | 0.734 |
| 1200 | 1300 | 26176 | 0.7 | 1214.8 | 1251.9 | 0.689 | 0.559 |
| 1300 | 1400 | 30036 | -1.2 | 930.2 | 1348.5 | 0.506 | 0.538 |
| 1400 | 1500 | 23156 | 1.4 | 1165.4 | 1448.3 | 0.451 | 0.234 |
| 1500 | 1600 | 17911 | 1.6 | 1557.4 | 1548.4 | 0.368 | 0.058 |
| 1600 | 1700 | 15296 | 3.1 | 1671.9 | 1648.0 | 0.272 | 0.443 |
| 1700 | 1800 | 12868 | 7.3 | 1610.2 | 1749.1 | 0.238 | 0.129 |
| 1800 | 1900 | 11550 | 10.5 | 1842.9 | 1849.7 | 0.219 | 0.061 |
| 1900 | 2000 | 12138 | 12.3 | 1683.3 | 1951.7 | 0.183 | 0.183 |
| 2000 | 2100 | 13163 | 12.2 | 1638.8 | 2049.5 | 0.171 | 0.173 |
| 2100 | 2200 | 10544 | 19.1 | 1886.8 | 2146.8 | 0.153 | 0.058 |
| 2200 | 2300 | 8208 | 33.7 | 2766.9 | 2247.9 | 0.091 | 0.183 |
| 2300 | 2400 | 4939 | 37.4 | 3644.2 | 2346.1 | 0.078 | 0.0 |
| 2400 | 2500 | 4006 | 42.5 | 4236.7 | 2450.8 | 0.062 | 0.0 |
| 2500 | 2600 | 3547 | 50.8 | 5041.9 | 2547.8 | 0.068 | 0.0 |
| 2600 | 2700 | 2661 | 55.4 | 5229.0 | 2647.4 | 0.075 | 0.061 |
| 2700 | 2800 | 2150 | 57.5 | 6384.3 | 2748.3 | 0.044 | 0.058 |
| 2800 | 2900 | 1721 | 58.3 | 7472.7 | 2846.7 | 0.033 | 0.067 |
| 2900 | 3000 | 1331 | 74.4 | 8846.3 | 2947.7 | 0.035 | 0.058 |
| 3000 | 3100 | 1098 | 78.4 | 10269.7 | 3048.9 | 0.037 | 0.067 |
| 3100 | 3200 | 869 | 87.3 | 11715.8 | 3147.4 | 0.030 | 0.0 |
| 3200 | 3300 | 771 | 88.8 | 11760.3 | 3249.9 | 0.022 | 0.0 |
| 3300 | 3400 | 654 | 94.5 | 13293.2 | 3348.9 | 0.033 | 0.0 |
| 3400 | 3500 | 596 | 80.5 | 11545.6 | 3449.6 | 0.026 | 0.125 |
| 3500 | 3600 | 362 | 103.0 | 16509.2 | 3549.2 | 0.022 | 0.0 |
| 3600 | 3700 | 585 | 82.6 | 10158.4 | 3660.7 | 0.027 | 0.0 |
| 3700 | 3800 | 566 | 91.5 | 11237.4 | 3743.8 | 0.022 | 0.0 |
| 3800 | 3900 | 680 | 93.5 | 10670.7 | 3844.9 | 0.031 | 0.0 |
| 3900 | 4000 | 406 | 102.6 | 15102.7 | 3944.7 | 0.028 | 0.0 |
| 4000 | 4100 | 281 | 105.8 | 13003.0 | 4052.3 | 0.040 | 0.0 |
| 4100 | 4200 | 234 | 117.9 | 16503.9 | 4148.8 | 0.034 | 0.061 |
| 4200 | 4300 | 149 | 134.6 | 23227.6 | 4242.6 | 0.024 | 0.0 |
| 4300 | 4400 | 208 | 151.4 | 26309.9 | 4344.9 | 0.024 | 0.0 |
| 4400 | 4500 | 136 | 114.6 | 16198.7 | 4447.5 | 0.029 | 0.061 |
| 4500 | 4600 | 101 | 137.4 | 20794.3 | 4548.0 | 0.031 | 0.0 |
| 4600 | 4700 | 87 | 149.5 | 24966.3 | 4637.7 | 0.027 | 0.0 |
| 4700 | 4800 | 20 | 163.6 | 29051.9 | 4736.6 | 0.034 | 0.0 |
| 4800 | 4900 | 8 | 198.3 | 43451.1 | 4834.6 | 0.029 | 0.051 |
| 4900 | 5000 | 4 | 111.5 | 46485.9 | 4961.1 | 0.037 | 0.0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5000 | 5100 | 1 | 252.5 | 63756.2 | 5018.6 | 0.037 | 0.0 |
| 5100 | 5200 | 4 | 82.2 | 18374.4 | 5163.8 | 0.028 | 0.061 |
| 5200 | 5300 | 1 | 268.4 | 72038.6 | 5235.8 | 0.037 | 0.0 |
| 5300 | 5400 | 0 | 0.0 | 0.0 | 0.0 | 0.020 | 0.0 |
| 5400 | 5500 | 0 | 0.0 | 0.0 | 0.0 | 0.016 | 0.0 |
| 5500 | 5600 | 0 | 0.0 | 0.0 | 0.0 | 0.012 | 0.0 |
| 5700 | 5800 | 0 | 0.0 | 0.0 | 0.0 | 0.004 | 0.0 |
| 5800 | 5900 | 0 | 0.0 | 0.0 | 0.0 | 0.002 | 0.0 |
| 5900 | 6000 | 0 | 0.0 | 0.0 | 0.0 | 0.002 | 0.0 |
| 7000 | 7100 | 0 | 0.0 | 0.0 | 0.0 | 0.002 | 0.0 |
| 8900 | 9000 | 0 | 0.0 | 0.0 | 0.0 | 0.002 | 0.0 |

where $(C_0)_i$ is the variance for each of the elevation ranges and $P_i$ is the percentage of the earth's surface area having that elevation range as estimated from the $1^\circ$ mean elevation data. Values of $C_0$ as estimated from (25) using all the data, and data from just the positive and negative elevations are given in Table Five.

Table Five

Estimates of $C_0$

| Method | $C_0$ (mgal$^2$) |
|---|---|
| Kaula (1959) | 1201 |
| Table Three | 1447 |
| Equation (25), all data | 1795 |
| Equation (25), negative elevations | 1772 |
| Equation (25), positive elevations | 1860 |
| Based on all anomalies without consideration of elevation ranges | 1644 |

For our future needs we select the $C_0 = 1795$ mgal$^2$ as the best estimate. A truer value may even be larger than this as certain high variance values found in certain elevation ranges are not represented in the 1795 figures as our elevation data was not sufficiently detailed to tell us what percentage of the earth's surface lies within these elevation ranges. The 1795 value should be more reliable than the value of 1447 estimated from Table Three, as a certain smoothing has taken place in deriving the Table Three estimates. In addition, it was necessary to make assumptions on the shape of the covariance curve in deriving the values for Table Three.

## 6. Anomaly Degree Variance Modeling

At this point we will develop a model for the anomaly degree variance which in turn will prove of value in deriving a closed expression for the covariance function of the disturbing potential and other gravimetric quantities. The basic procedures for this modeling have been discussed by Rapp (1973a). However, we introduce for this paper the s term and the $\beta_\ell$ term.

We first postulate an anomaly degree variance model of the following form:

$$c_\ell = \frac{A(\ell - 1)}{(\ell - 2)(\ell + B)} \tag{25A}$$

This model had originally been suggested by Tscherning. Best estimates for the A and B parameters are to be found subject to the following data:

1. Anomaly Degree Variances Determined From Potential Coefficients

The values of $c_\ell$ that are used here are for $\ell = 3$ to 20 are those found from the least squares collocation solution for potential coefficients as described in Rapp (1973b). These values are given in Table Five.

Table Five

Anomaly Degree Variances From Potential Coefficients (Rapp, 1973b) ($mgal^2$)

| $\ell$ | $c_\ell$ | $\ell$ | $c_\ell$ |
|---|---|---|---|
| 3 | 33.9 | 12 | 4.8 |
| 4 | 19.2 | 13 | 11.7 |
| 5 | 21.6 | 14 | 5.5 |
| 6 | 18.9 | 15 | 7.3 |
| 7 | 18.8 | 16 | 6.5 |
| 8 | 10.4 | 17 | 5.7 |
| 9 | 11.1 | 18 | 10.7 |
| 10 | 11.4 | 19 | 11.0 |
| 11 | 8.4 | 20 | 8.9 |

No formal standard deviations were attached to these values of $c_\ell$.

These values of $c_\ell$ can be directly used with (25 A).

2. Anomaly Block and Point Variances

We have previously determined the block variances for $1^\circ$ and $5^\circ$ equal area blocks. These values can be related to $c_\ell$ values through equation (11) which is re-written for the variance (i.e. $\psi = 0$) as:

$$\overline{C}(\psi = 0) = \sum_{\ell=0}^{\infty} \beta_\ell^2 c_\ell s^{\ell+2} \tag{26}$$

Equation (26) is also valid for point anomalies recalling that in this case $\beta_\ell$ equals one.

In (26) the summation is started from $\ell = 0$ but in fact we are trying to model $c_\ell$ from degree 3. Thus, we carry out the summation to degree 3 but we must modify our point and block variances by essentially removing the $c_2$ value. From Rapp (1973b) $c_2 = 7.5$ $mgal^2$. The modified data is shown in Table Six.

Table Six

Modified* Point and Block Variances For Anomaly Degree Variance Fitting

| Size | Modified Variance |
|---|---|
| Point | 1788 $mgal^2$ |
| $1^\circ$ | 912 " |
| $5^\circ$ | 295 " |

*to refer to a complete second degree field

The adjustment procedure was carried out by first trying to determine best estimates of A and B for equation (25) by using the data of Table Five and the block variances of Table Six. The value of $\beta_\ell$ needed in (26) was computed using a $\psi$ value determined from equation (13). Tests indicated the summation to $\infty$ in (26) could safely be replaced by a summation to (4) $(180^\circ)/\theta^\circ$ or to $720/\theta^\circ$. Various runs were made with different s values to determine a proper value such that the summation to $\infty$ (or in practice a high number such as 50,000 or 100,000) would come close to the modified point variance of 1788 $mgal^2$. (It was found that for an accuracy of 0.1 mgals it was sufficient to carry out the point anomaly summation to $\ell = 16000$ while for a 0.001 mgal accuracy the summation should be carried to about $\ell = 30000$).

For theoretical reasons to be seen later, the B unknown in equation (25A) should be an integer. To produce such an unknown we first made an adjustment letting A and B adjust freely. The resultant B found was 24.03. We then repeated the adjustment, fixing B at 24 exactly. In this adjustment the two block variances were given weights of 1/100. All anomaly degree variances except for degree 3 and 4 were given weights of 1/.64. At degree 3 a weight of 1/.08 was used while at degree 4 a weight of 1/.16 was used. These weight assignments were made only to assure a reasonable fit to the data and were not based on relative accuracy considerations of the data.

We give in Table Seven the parameters of the final model.

Table Seven

Parameters of Anomaly Degree Variance Model

$A = 425.28$ $mgal^2$
$B = 24$ (exact)
$s = 0.999617$

We give in Table Eight a comparison of the anomaly degree variances from Table Five and those as computed from Equation (25A) using the A and B values given in Table Seven.

Table Eight

Anomaly Degree Variances ($mgal^2$)

| | Original Table 5 | Equation (25A) | | Original | Equation (25A) |
|---|---|---|---|---|---|
| 3 | 33.9 | 31.5 | 12 | 4.8 | 13.0 |
| 4 | 19.2 | 22.8 | 13 | 11.7 | 12.5 |
| 5 | 21.6 | 19.6 | 14 | 5.5 | 12.1 |
| 6 | 18.9 | 17.7 | 15 | 7.3 | 11.7 |
| 7 | 18.8 | 16.5 | 16 | 6.5 | 11.4 |
| 8 | 10.4 | 15.5 | 17 | 5.7 | 11.1 |
| 9 | 11.1 | 14.7 | 18 | 10.7 | 10.8 |
| 10 | 11.4 | 14.1 | 19 | 11.0 | 10.5 |
| 11 | 8.4 | 13.5 | 20 | 8.9 | 10.2 |

The root mean square difference between the original and adjusted values was $\pm 4.0$ $mgal^2$. The $1^\circ$ residual block variance from the adjusted model is 841 $mgal^2$ with the $5^\circ$ residual block variance being 360 $mgal^2$ as compared to the corresponding values of 912 $mgal^2$ and 295 $mgal^2$ as given in Table Six. By summing (26) with $\beta_\ell = 1$ to a sufficiently high degree (50000) the point variance implied by this model is 1788 $mgal^2$. If we wished, at this point, the covariance functions implied by this new anomaly degree variance model could be computed by substitution of the model into equation (10) or (11). This type of computation will be postponed until the discussion of the closed covariance function expressions.

## 7. Relationship Between the Covariance Function of the Anomalous Potential and Covariance Functions of Gravity Anomalies or Deflections of the Vertical

As explained e.g. in Moritz (1972, p. 97), covariance functions of quantities related to the anomalous potential can be derived from the covariance function of the anomalous potential K(P, Q). The covariance between two quantities A and B, derived by applying a certain operation on T can be found by applying the same operation on K(P, Q). Moritz calls this fact "the law of propagation of covariances". We have above used the law to derive (15), and thereby the relation between K(P, Q) and C(P, Q). In the following we will derive the relationship between K(P, Q) and the covariances of or between the height anomaly $\zeta$, the free-air gravity anomaly $\Delta g$ and the two deflection components $\xi$ and $\eta$.

We will use the same notation for the covariance functions as used in Moritz (1972), i.e. cov(A, B) for the covariance of the two quantities A and B. The relationship between the gravity anomaly and the anomalous potential is given above in (14). For the three other quantities we have the well known relations:

$$\zeta = T/\gamma\ , \tag{27}$$

$$\xi = -\frac{1}{\gamma \cdot r} \cdot \frac{\partial T}{\partial \varphi} \quad \text{and} \tag{28}$$

$$\eta = -\frac{1}{\cos\varphi \cdot \gamma \cdot r} \cdot \frac{\partial T}{\partial \lambda}\ , \tag{29}$$

where $\gamma$ is the reference gravity, r the distance from the center of the Earth, $\varphi$ the latitude and $\lambda$ the longitude. It will for most purposes be sufficient to work in spherical approximation. But we will not restrict ourselves to consider only points on the surface of the Earth.

On the surface of the Earth r is substituted by a mean Earth radius ($R_e$), $\gamma$ by a mean gravity value (G), and $\varphi$ by the geocentric latitude. For a point outside (or inside) the surface of the Earth, we will substitute for r the radius of a sphere e.g. including the same volume as an ellipsoid confocal with the adopted reference ellipsoid and passing through the considered point. (Thus, we will still call this quantity r). The reference gravity can then be substituted by $kM/r^3$ and $\varphi$ again with the proper geocentric latitude. (In practice $\varphi$ is just treated as if it was equal to the geocentric latitude).

We will introduce a more compact notation for the partial derivative with respect to an independent variable e.g. r:

$$D_r = \frac{\partial}{\partial r}\ ,$$

and for the second order partial derivative with respect to r and t:

$$D^2_{rt} = \frac{\partial^2}{\partial r \partial t}$$

The formulae (27), (14), (28) and (29) becomes then:

$$\zeta = T/G \tag{30}$$

$$\Delta g = -D_r T - \frac{2}{r}\, T \tag{31}$$

$$\xi = -\frac{1}{G \cdot r}\, D_\varphi T \quad \text{and} \tag{32}$$

$$\eta = -\frac{1}{G \cdot r \cdot \cos\varphi}\, D_\lambda T. \tag{33}$$

Using the law of propagation of covariances given by Moritz (1972, p. 97) applied to equations (30) - (33) we find:

$$\operatorname{cov}(T_P,\ T_Q) = K(P,Q) \tag{34}$$

$$\operatorname{cov}(\Delta g_P, \Delta g_Q) = C(P,Q) = D_r D_{r'} K(P,Q) + \frac{2}{r}\cdot D_{r'} K(P,Q) + \frac{2}{r'} D_r K(P,Q) + \frac{4}{rr'} K(P,Q), \tag{35}$$

$$\operatorname{cov}(\Delta g_P, \zeta_Q) = \left(-D_r K(P,Q) - \frac{2}{r} K(P,Q)\right)\cdot \frac{1}{G'}, \tag{36}$$

$$\operatorname{cov}(\zeta_P, \zeta_Q) = K(P,Q)/(G\cdot G'), \tag{37}$$

$$\operatorname{cov}(\xi_P, \zeta_Q) = -D_\varphi K(P,Q)/(G\cdot G'\cdot r), \tag{38}$$

$$\operatorname{cov}(\eta_P, \zeta_Q) = -D_\lambda K(P,Q)/(G' G\cdot r\cdot \cos\varphi) \tag{39}$$

$$\operatorname{cov}(\xi_P, \xi_Q) = D_\varphi D_{\varphi'} K(P,Q)/(G'\cdot G\cdot r\cdot r') = D^2_{\varphi\varphi'} K(P,Q)/(G\cdot G'\cdot rr'), \tag{40}$$

$$\operatorname{cov}(\xi_P, \eta_Q) = D^2_{\varphi\chi} K(P,Q)/(G'\cdot r'\cdot \cos\varphi'\cdot r\cdot G), \tag{41}$$

$$\operatorname{cov}(\eta_P, \eta_Q) = D^2_{\lambda\chi'} K(P,Q)/(G'\cdot G\cdot rr' \cos\varphi\cdot \cos\varphi'), \tag{42}$$

$$\operatorname{cov}(\Delta g_P, \xi_Q) = -D_{\varphi'}(\operatorname{cov}(\Delta g_P,\ \zeta_Q))/r' = D_{\varphi'}\left(D_r K(P,Q) + \frac{2}{r} K(P,Q)\right)/(G' r') \tag{43}$$

$$\operatorname{cov}(\Delta g_P, \eta_Q) = -D_{\chi'}(\operatorname{cov}(\Delta g_P, \zeta_Q))/(r'\cdot \cos\varphi') = D_{\lambda'}\left(D_r K(P,Q) + \frac{2}{r} K(P,Q)\right)/(G' r' \cos\varphi'), \tag{44}$$

where the quantities marked with an apostrophe refer to Q and the unmarked quantities refer to P.

The covariances involving the deflections components ((38) - (44)) are most easily expressed (and computed) as derivatives with respect to the cosine of the spherical distance $\psi$ between P and Q. (We will from now on only regard isotropic covariance functions K(P, Q), i.e. so that (9) always is valid and hence K(P, Q) only depends on $\psi$, r and r').

Putting $t = \cos\psi$, $D_t K(P,Q) = K'$ and $D_t^2 K(P,Q) = K''$ we get:

$$D_\varphi K = D_\varphi t \cdot K'$$

$$D_\lambda K = D_\lambda t \cdot K'.$$

Hence

$$D^2_{\varphi\varphi'} K = D_\varphi t \cdot D_{\varphi'} t \cdot K'' + D^2_{\varphi\varphi'} t \cdot K' \tag{45}$$

$$D^2_{\varphi\lambda'} K = D_\varphi t \cdot D_{\lambda'} t \cdot K'' + D^2_{\varphi\lambda'} t \cdot K') \tag{46}$$

$$D^2_{\lambda\lambda'} K = D_\lambda t \cdot D_{\lambda'} t \cdot K'' + D^2_{\lambda\lambda'} t \cdot K'. \tag{47}$$

$$-D_{\varphi'} (\mathrm{cov}(\Delta g_P, T_Q)) = D_{\varphi'} t \cdot (D^2_{rt} K(P,Q) + \frac{2}{r} D_t K(P,Q)) \tag{48}$$

$$-D_{\lambda'} (\mathrm{cov}(\Delta g_P, T_Q)) = D_{\lambda'} t \cdot (D^2_{rt} K(P,Q) + \frac{2}{r} D_t K(P,Q)) \tag{49}$$

Note, the common factors $K'$ and $K''$ in (47), (48) and (49), i.e., the three covariance functions $\mathrm{cov}(\xi_P, \xi_Q)$, $\mathrm{cov}(\xi_P, \eta_Q)$ and $\mathrm{cov}(\eta_P, \eta_Q)$ can easily be computed at the same time. The covariance functions (38) - (44) are used in actual prediction computations involving deflections either as observed quantities or as quantities to be predicted. These covariance functions are not anymore isotropic. Then for theoretical discussions it is more convenient to regard the covariances, where one or both of the quantities are either the longitudinal ($\ell$) or the transverse component (m) of the deflection of the vertical. This type of covariance function will be isotropic and will have a simple relation to K(P, Q).

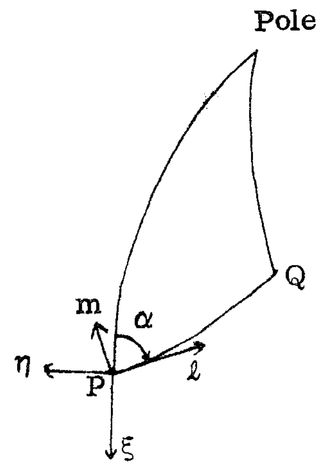



Figure 2.

Spherical triangle (Pole, Q, P) with the deflection components $(\xi, \eta)$ and $(\ell, m)$ shown as vectors.

In Moritz (1972) the relationships between K(P, Q) and the covariance functions are expressed in terms of derivatives with respect to $\psi$. We will express the relations in terms of derivatives with respect to $t = \cos\psi$.

Let the azimuth between P and Q be $\alpha$. Then we have (cf. figure 2):

$$\ell_P = \cos\alpha \cdot (-\xi_P) + \sin\alpha \cdot (-\eta_P) \text{ and}$$
$$m_P = \sin\alpha \cdot (-\xi_P) - \cos\alpha \cdot (-\eta_P) \tag{50}$$

Using (38) and (39) and the law of propagation of covariances, we get:

$$\mathrm{cov}(\ell_P, \zeta_Q) = (\cos\alpha \cdot D_\varphi t \cdot K' + \sin\alpha \cdot D_\lambda t \cdot K' \cdot \frac{1}{\cos\varphi})/(G \cdot G' \cdot r) \text{ and}$$

$$\mathrm{cov}(m_P, \zeta_Q) = (\sin\alpha \cdot D_\varphi t \cdot K' - \cos\alpha \cdot D_\lambda t \cdot K' \cdot \frac{1}{\cos\varphi})/(G \cdot G' \cdot r).$$

Because

$$t = \sin\varphi \cdot \sin\varphi' + \cos\varphi \cdot \cos\varphi' \cdot \cos(\lambda' - \lambda)$$

we have

$$D_\varphi t = \cos\varphi \cdot \sin\varphi' - \sin\varphi \cdot \cos\varphi' \cdot \cos(\lambda' - \lambda) = \sin\psi \cdot \cos\alpha \text{ and}$$
$$D_\lambda t = \cos\varphi \cdot \cos\varphi' \sin(\lambda' - \lambda) = \cos\varphi \cdot \sin\psi \cdot \sin\alpha,$$

hence

$$\mathrm{cov}(\ell_P, \zeta_Q) = \sin\psi \cdot K'/(G \cdot G' \cdot r) \text{ and} \tag{51}$$

$$\mathrm{cov}(m_P, \zeta_Q) = 0 \tag{52}$$

For the covariance with the gravity anomaly we get in the same way (using the law of propagation of covariations and (14))

$$\mathrm{cov}(\ell_P, \Delta g_Q) = +(D_r\ \ K' + \frac{2}{r} K') \cdot \sin\psi/(G \cdot r) \tag{53}$$

$$\mathrm{cov}(m_P, \Delta g_Q) = 0 \tag{54}$$

The expressions for $\mathrm{cov}(\ell_P, \ell_Q)$, $\mathrm{cov}(\ell_P, m_Q)$ and $\mathrm{cov}(m_P, m_Q)$ are derived in a very simple way in Moritz (1972, p. 109). We repeat the results expressed as derivatives of t.

$$\mathrm{cov}(\ell_P, \ell_Q) = -D_\psi^2 K/(G \cdot G' \cdot r \cdot r') = (t \cdot K' - \sin^2\psi \cdot K'')/(G \cdot G' \cdot r \cdot r') \tag{55}$$

$$\mathrm{cov}(\ell_P, m_Q) = 0 \quad \text{and} \tag{56}$$

$$\mathrm{cov}(m_P, m_Q) = -D_\psi K/(\sin\psi \cdot G \cdot G' \cdot r \cdot r') = K'/(G \cdot G' \cdot r \cdot r'). \tag{57}$$

From the formulae (51) - (57) several intersting consequences of the imposed isotropic property can be seen. The deflection components at P are independent of the height anomaly and the gravity anomaly in P. The transverse component of the deflection in P, $m_P$ is independent of $\zeta_Q$, $\Delta g_Q$ and $\ell_Q$. For $\xi_P$ this implies, that $\xi_P$ is independent of $\eta_Q$ for $\varphi = \varphi'$ and $\eta_P$ independent of $\xi_Q$ for $\lambda = \lambda'$.

Finally we will conclude that the basic quantities to be computed in the evaluation of the expressions (34) - (49) and (51) - (57) are K, $K'$, $K''$, $D_r K' + \frac{2}{r} K$ and $\mathrm{cov}(\Delta g_P, \Delta g_Q)$.

## 8. Closed covariance function expressions.

In this section we will consider different models for the degree-variances and explain how closed expressions for corresponding covariance functions can be obtained. We will distinguish between different types of degree-variances and hence between different covariance functions models. Thus we will still consider only isotropic models. A subscript k will be used to distinguish between the models. Then we can define $\sigma_{k,\ell}(A, B)$ to be the degree-variances of degree $\ell$ in the k'th degree-variance model, i.e. so that the corresponding covariance function becomes:

$$\mathrm{cov}_k(A, B) = \left(\frac{R}{r}\right)^{I} \left(\frac{R}{r'}\right)^{J} \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(A, B)\, s^{\ell+1} P_\ell(t)\ , \tag{58}$$

where I and J are either 0 or 1. (Note, that for I=J=1 we have $\left(\frac{R}{r}\right)\cdot\left(\frac{R}{r'}\right) = s$).

For the already introduced quantities $c_\ell$ and $\sigma_\ell$ we then have:

$$c_\ell = \sigma_{k,\ell}(\Delta g, \Delta g) \text{ and}$$

$$\sigma_\ell = \sigma_{k,\ell}(T, T).$$

The corresponding covariance functions become, using (9) and (10)

$$\operatorname{cov}_k(T_P, T_Q) = \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) s^{\ell+1} P_\ell(t) \text{ and} \tag{59}$$

$$\begin{aligned} \operatorname{cov}_k(\Delta g_P, \Delta g_Q) &= \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(\Delta g, \Delta g) s^{\ell+2} P_\ell(t) \\ &= \left(\frac{R}{r}\right)\left(\frac{R}{r'}\right) \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(\Delta g, \Delta g) s^{\ell+1} P_\ell(t). \end{aligned} \tag{60}$$

The relationship (15) becomes:

$$\sigma_{k,\ell}(T, T) = \frac{R^2}{(\ell-1)^2} \sigma_{k,\ell}(\Delta g, \Delta g) \tag{61}$$

In the following we will also consider the degree-variances $\sigma_{k,\ell}(\Delta g, T)$ of the covariance function $\operatorname{cov}_k(\Delta g_P, T_Q)$ which is related to the covariance (36) by:

$$\operatorname{cov}_k(\Delta g_P, T_Q) = \operatorname{cov}(\Delta g_P, \zeta_Q)\cdot G'$$

Using (36) and (59) we get:

$$\begin{aligned} \operatorname{cov}_k(\Delta g_P, T_Q) &= -D_r\left(\sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) s^{\ell+1} P_\ell(t)\right) - \frac{2}{r}\left(\sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) s^{\ell+1} P_\ell(t)\right) \\ &= \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) \frac{(\ell-1)}{r} s^{\ell+1} P_\ell(t) \\ &= \frac{R}{r} \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T, T) \frac{(\ell-1)}{R} s^{\ell+1} P_\ell(t). \end{aligned} \tag{62}$$

Hence, using (58) we see that I = 1 and J = 0 and that

$$\sigma_{k,\ell}(\Delta g, T) = \sigma_{k,\ell}(T, T)\cdot\frac{(\ell-1)}{R} \qquad \text{and} \tag{63}$$

$$\mathrm{cov}_k(\Delta g_P, T_Q) = \frac{R}{r}\sum_{\ell=0}^{\infty} \sigma_{k,\ell}(\Delta g, T)s^{\ell+1}P_\ell(t). \tag{64}$$

(Note, that the introduced notation can't be used for covariance-functions involving deflections. These covariance functions can be expressed as the sums of series in $P'_\ell(t)$ and $P''_\ell(t)$ (apostrophe mean differentiation with respect to t), and not on the form (58) as a series in $P_\ell(t)$ and $s^{\ell+1}$.)

Five different models of the anomaly degree variances will be discussed below, i.e., k will take on values 1, 2, ... 5.

In Tscherning (1972), analytic models have been described for covariance function having anomaly degree-variances equal to:

$$\sigma_{1,\ell}(\Delta g, \Delta g) = A_1(\ell-1)^2, \quad \ell>1 \tag{65}$$

$$\sigma_{2,\ell}(\Delta g, \Delta g) = A_2(\ell-1)/\ell, \quad \ell>1 \text{ and} \tag{66}$$

$$\sigma_{3,\ell}(\Delta g, \Delta g) = A_3(\ell-1)/(\ell-2), \quad \ell>2, \tag{67}$$

where $A_1$, $A_2$, and $A_3$ ( and below $A_4$ and $A_5$) are positive constants of dimension mgal$^2$. These types of models have been further considered by Rapp (1972a).

$$\sigma_{4,\ell}(\Delta g, \Delta g) = A_4\,\frac{(\ell-1)}{(\ell-2)(\ell+B)} \quad \text{and} \tag{68}$$

$$\sigma_{5,\ell}(\Delta g, \Delta g) = A_5\,\frac{(\ell-1)}{(\ell-2)\,(\ell+B+\beta\ell^2)}, \quad \ell>2 \tag{69}$$

For $i+j = \frac{1}{\beta}$ and $i\cdot j = B/\beta$ we can write (69):

$$\sigma_{5,\ell}(\Delta g, \Delta g) = \frac{A_5}{\beta}\cdot\frac{(\ell-1)}{(\ell-2)(\ell+i)(\ell+j)} \quad . \tag{70}$$

As indicated above, the covariance functions corresponding to models 1, 2 and 3 can be represented by closed expressions. (By closed expression we mean expressions which only contain a finite number of terms). This is also true for model 4 and 5, provided we place some restrictions on B or i and j. First of all the resulting degree-variances have to be greater than or equal to zero for $\ell$ greater than 2. Hence, B and i, j will have to be greater than -2. And the technique used below for the derivation will imply that we have to restrict B and i, j to integer values and that we also will have to require that i is unequal to j and that all three quantities are greater than -1.

We will not consider the covariance functions derived using the model (65) because the anomaly degree-variances are unrealistic. Thus, the model leads to very simple closed expressions for the covariance functions, which can be found, e.g. in Tscherning (1972).

The technique we will use for the derivation of the closed covariance expressions is very simple. The covariance functions can be split into components which, upon multiplication by appropriate constants will yield the covariance function. These components can be expressed as:

$$F = \sum_{\ell=0}^{\infty} s^{\ell+1} P_\ell(t) \text{ and} \tag{71}$$

$$F_i = \sum_{\ell=0}^{\infty} \frac{1}{\ell+i} s^{\ell+1} P_\ell(t) \quad \text{for } i>0 \tag{72}$$

$$F_i = \sum_{\ell=-i+1}^{\infty} \frac{1}{\ell+i} s^{\ell+1} P_\ell(t) \quad \text{for } i \leq 0, \quad \text{and} \tag{73}$$

as the first and second derivatives of F or $F_i$ with respect to t,

$$F', F'', F_i', F_i''.$$

We have for example (using (59) (61) and (67)):

$$\mathrm{cov}_3(T_P, T_Q) = A_3 \sum_{\ell=3}^{\infty} \frac{R^2}{(\ell-1)(\ell-2)} s^{\ell+1} P_\ell(t)$$

$$= A_3 \cdot R^2 \sum_{\ell=3}^{\infty} \left(\frac{1}{\ell-2} - \frac{1}{\ell-1}\right) s^{\ell+1} P_\ell(t)$$

$$= A_3 \cdot R^2 (F_{-2} - (F_{-1} - s^3 P_2(t))).$$

The closed expression for the function F can be derived using the well known formula (Heiskanen and Moritz, 1967, eq. 1-80):

$$F = s \cdot \sum_{\ell=0}^{\infty} s^{\ell} P_{\ell}(t) = \frac{s}{\sqrt{1-2st+s^2}} . \tag{74}$$

The denominator will be one of the basic quantities in the following derivations, so we will use:

$$\begin{aligned} L &= \sqrt{1-2st+s^2} , \\ M &= 1 - L - s \cdot t \quad \text{and} \\ N &= 1 + L - st . \end{aligned} \tag{75}$$

We then have:

$$F = \frac{s}{L} .$$

The functions $F_i$ can be derived by multiplying $F$ or $\frac{1}{L}$ by an appropriate power of s and integrating the expression with respect to s.

Using:

$$\int_0 s^{\ell+i-1} \, ds = \frac{s^{\ell+i}}{\ell+i} , \quad \ell + i > 0$$

we see, that by integrating

$$\frac{s^{i-1}}{L} = \sum_{\ell=0}^{\infty} s^{\ell+i-1} P_{\ell}(t) \tag{76}$$

we should be able to find $F_i$. We have by (72)

$$s^{i-1} F_i = \sum_{\ell=0}^{\infty} \frac{s^{\ell+i}}{\ell+i} P_{\ell}(t) \text{ for } i>0 \text{ and by (73):} \tag{77}$$

$$s^{i-1} F_i = \sum_{\ell=0,\ell\neq -i}^{\infty} \frac{s^{\ell+i}}{\ell+i} P_\ell(t) - \sum_{\ell=0}^{i-1} \frac{s^{\ell+i}}{\ell+i} P_\ell(t), \quad i \leq 0 . \tag{78}$$

The integrals:

$$\int \frac{s^i}{L}\, ds, \quad i = -2, -1, 0, 1, 2 \tag{79}$$

can be found in integral tables as Gradshteyn-Ryzhik (abbreviated below to G. R.), (1965).

From these basic integrals, $F_i$ can be computed using recursion formulae. We will first consider negative powers of s.

Using G.R. 2.268 we get:

$$\int \frac{ds}{s^i L} = -\frac{L}{(i-1)s^{i-1}} + \frac{(2i-3)\cdot t}{(i-1)} \int \frac{ds}{s^{i-1}L} - \frac{i-2}{(i-1)} \int \frac{ds}{s^{i-2}L}, \quad i>0, \tag{80}$$

and hence

$$\int \frac{ds}{s^2 L} = -\frac{L}{s} + t \cdot \int \frac{ds}{s \cdot L} + a_{-2} \tag{81}$$

$$\begin{aligned} \int \frac{ds}{s^3 L} &= -\frac{L}{2s^2} + \frac{3}{2} t \left( -\frac{L}{s} + t \int \frac{ds}{s \cdot L} \right) - \frac{1}{2} \int \frac{ds}{s \cdot L} + a_{-3} \\ &= -\frac{3ts+1}{2s^2} \cdot L + P_2(t) \int \frac{ds}{s \cdot L} + a_{-3} \end{aligned} \tag{82}$$

where $a_{-2}$, $a_{-3}$ are integration constants. From G.R. 2.266 we have:

$$\begin{aligned} \int \frac{ds}{s \cdot L} &= -\ln \frac{2 - 2ts + 2 \cdot L}{s} + a_{-1} = \ln \frac{2}{1 - ts + L} + \ln(s) + \ln(4) + a_{-1} \\ &= \ln \frac{2}{N} + \ln(s) + \ln(4) + a_{-1} \end{aligned} \tag{83}$$

The constant $a_{-1}$ is determined requiring (78) to be zero for s equal to zero.

$$s^{-1}F_0 = \int \frac{ds}{s \cdot L} - \int \frac{ds}{s} = \ell n \frac{2}{N} + \ell n (4) + a_{-1} ,$$

hence $a_{-1} = -\ell n(4)$ and then:

$$s^{-1} F_0 = \ell n \frac{2}{N} . \tag{84}$$

Then we can compute $F_{-1}$ and $F_{-2}$.

$$s^{-2} F_{-1} = \sum_{\ell=2}^{\infty} \frac{s^{\ell-1}}{\ell-1} P_\ell(t) = -\frac{L}{s} + t \cdot \ell n \frac{2}{N} - \int \frac{ds}{s^2} + a_{-2}$$

$$= \frac{1-L}{s} + t \cdot \ell n \frac{2}{N} + a_{-2} = \frac{1-ts-L}{s} + t \cdot \ell n \frac{2}{N} \quad \text{or} \tag{85}$$

$$F_{-1} = s \cdot (M + ts \cdot \ell n \frac{2}{N}), \tag{86}$$

$$s^{-3} F_{-2} = \sum_{\ell=3}^{\infty} \frac{s^{\ell-2}}{\ell-2} P_\ell(t) = \int \left( \frac{1}{s^3} + \frac{t}{s^2} - \frac{P_2(t)}{s} \right) ds + \int \frac{ds}{s^3 L} + a_{-3}$$

$$= \frac{1}{2s^2} + \frac{t}{s} - P_2(t) \cdot \ell n(s) + (P_2(t) \cdot (\ell n \frac{2}{N} + \ell n(s)) - \frac{3ts+1}{2s^2} L + a_{-3})$$

$$= (1 + 2ts - (3ts+1) \cdot L)/(2s^2) + P_2(t) \cdot \ell n \frac{2}{N} + a_{-3}.$$

The constant $a_{-3}$ can now be determined. Because $\ell n \frac{2}{N}$ is zero for s equal to zero, we must have:

$$\lim_{s \to 0} \frac{1 + 2ts - (3ts+1) \cdot L}{2s^2} = -a_{-3}$$

The limit can be determined using the rule of l'Hospital two times. Note first, that $D_s L = (s-t)/L$. We then get

$$\lim_{s\to 0} \frac{1+2st-(3ts+1)\cdot L}{2s^2} = \lim_{s\to 0} \frac{-3t\cdot L-(3ts+1)(s-t)/L+2t}{4s}$$

$$= \lim_{s\to 0} (-3t(s-t)/L-(6ts+1-3t^2)/L+(3ts^2+s-3t^2s-t)(s-t)/L^3)/4$$

and hence:

$$a_{-2} = -\frac{7t^2-1}{4} = -\frac{3}{2}t^2+\frac{1-t^2}{4}.$$

We then get:

$$F_{-2} = s((-3t^2s^2+2ts+1-(3ts+1)\cdot L)/2+(P_2(t)\cdot \ln\frac{2}{N}+\frac{1-t^2}{4})\cdot s^2)$$

$$= s((1-ts-L)(3ts+1)/2+s^2(P_2(t)\cdot \ln\frac{2}{N}+\frac{1-t^2}{4})) \tag{87}$$

$$= s(M\cdot(3ts+1)/2+s^2(P_2(t)\cdot \ln\frac{2}{N}+(1-t^2)/4)).$$

For the evaluation of covariance functions involving deflections, we have to compute $F_0'$, $F_0''$, $F_{-1}'$, $F_{-1}''$, $F_{-2}'$, and $F_{-2}''$ (where again the apostrophe means differentiation with respect to $t=\cos\psi$).

We will first compute some auxiliary quantities:

$$D_t L = -\frac{s}{L}, \quad D_t(\frac{1}{L}) = \frac{s}{L^3},$$

$$D_t M = -s+\frac{s}{L} = s\frac{(1-L)}{L}$$

$$D_t N = -s-\frac{s}{L} = -s(1+L)/L.$$

Hence from (84) we get by differentiation:

$$F_0' = s \cdot D_t \ln \frac{2}{N} = \frac{s^2(1+L)}{L \cdot N} = s^2 \left(\frac{1}{L \cdot N} + \frac{1}{N}\right) \tag{88}$$

$$F_0'' = s^2\left(\frac{-N(-s/L) - L(-s(1+L)/L)}{(L \cdot N)^2} + \frac{s(1+L)/L}{N^2}\right) \tag{89}$$

$$= s^3\left(\frac{N+L(1+L)}{L^3 \cdot N^2} + \frac{1+L}{L \cdot N^2}\right) = s^3\left(\frac{N+L}{L^3 \cdot N^2} + \frac{2+L}{L \cdot N^2}\right)$$

For $F_{-1}$ we get using (86)

$$\begin{aligned} F_{-1}' &= s(D_t M + F_0 + t \cdot F_0') \\ &= s(s(1-L)/L + F_0 + t \cdot s^2 \left(\frac{1}{L \cdot N} + \frac{1}{N}\right)) \\ &= s^2 \left((1-L)/L + \ln \frac{2}{N} + t \cdot s\left(\frac{1}{L \cdot N} + \frac{1}{N}\right)\right), \end{aligned} \tag{90}$$

$$\begin{aligned} F_{-1}'' &= s(D_t^2 M + 2F_0' + t \cdot F_0'') \\ &= s\left(\frac{s^2}{L^3} + 2s^2\left(\frac{1}{L \cdot N} + \frac{1}{N}\right) + t \cdot s^3\left(\frac{N+L}{L^3 \cdot N^2} + \frac{2+L}{L \cdot N^2}\right)\right) \\ &= s^3\left(\frac{1}{L^3} + \frac{2(1+L)}{L \cdot N} + ts\left(\frac{1}{L^3 \cdot N} + \frac{(1+L)^2}{(L \cdot N)^2}\right)\right) \end{aligned} \tag{91}$$

and for $F_{-2}$ we get by (87)

$$\begin{aligned} F_{-2}' &= s\,[\,D_t M \cdot (3ts+1)/2 + M \cdot \frac{3s}{2} + s^2 (3t \cdot \ln \frac{2}{N} + B_2(t) \cdot D_t \ln \frac{2}{N} - t/2)] \\ &= s^2[(3ts+1)(1-L)/(2L) + \frac{3}{2} M + s(3t \cdot \ln \frac{2}{N} + B_2(t) \cdot s\left(\frac{1}{LN} + \frac{1}{N}\right) - t/2)] \\ &= s^2[\tfrac{1}{2}\left((3ts+1)/L + 2 - 7ts - 3L\right) + s\left(3t \cdot \ln \frac{2}{N} + B_2(t) \cdot s\left(\frac{1+L}{L \cdot N}\right)\right)]. \end{aligned} \tag{92}$$

$$F''_{-2} = s^2\left[\tfrac{1}{2}\left(\frac{L\cdot 3s-(3ts+1)(-s/L)}{L^2} - 7s+\frac{3s}{L}\right)+s\left(3\ln\frac{2}{N}+6ts\frac{1+L}{L\cdot N}\right.\right.$$

$$\left.\left.+P_2(t)\cdot s\left(-\frac{-s(1+L)}{N^2\cdot L}-\frac{(-s/L)\cdot N-L(s(1+L)/L)}{(L\cdot N)^2}\right)\right)\right] \tag{93}$$

$$= s^3\left[\left(\frac{6}{L}+\frac{3ts+1}{L^3}-7\right)\tfrac{1}{2}+3\ln\frac{2}{N}+6ts\,\frac{1+L}{L\cdot N}+P_2(t)\cdot s^2\left(\left(\frac{1+L}{L\cdot N}\right)^2+\frac{1}{L^3 N}\right)\right].$$

The closed expressions for $F_i$, $i > 0$ can be found using another recursion formula, G.R. 2.263. We will treat this case in a more general way, because in this case we want to derive expressions not only for $i = 1$, 2 and 3 but for $i = 1$ to $\infty$. It is also necessary to have a recursion formula well suited for actual computations.

We have (using G.R. 2.263):

$$\int \frac{s^i ds}{L} = \frac{s^{i-1}\cdot L}{i} + \frac{(2i-1)t}{i}\int \frac{s^{i-1}}{L}\,ds - \frac{(i-1)}{i}\int \frac{s^{i-2}}{L}\,ds \quad \text{or} \tag{94}$$

$$\frac{1}{s^{i-1}}\int \frac{s^i ds}{L} = \left(L+(2i-1)t\cdot\frac{1}{s^{i-1}}\int\frac{s^{i-1}ds}{L} - \frac{(i-1)}{s}\cdot\frac{1}{s^{i-2}}\int\frac{s^{i-2}}{L}ds\right)\cdot\frac{1}{i} \tag{95}$$

Realizing that:

$$F_i = \frac{1}{s^{i-1}}\int_0 \frac{s^{i-1}}{L}\,ds \qquad \text{for } i>0$$

we get

$$F_{i+1} = \left(L+(2i-1)t\cdot F_i - \frac{(i-1)}{s}\cdot F_{i-1}\right)\cdot\frac{1}{s\cdot i}\,. \tag{96}$$

Fortunately we can use the recursion formula for the computation of $D_t F_i = F'_i$ and $D_t^2 F_i = F''_i$ as well.

Differentiating (96) we have:

$$F'_{i+1} = \left(D_t L+(2i-1)(F_i+t\cdot F'_i) - \frac{(i-1)}{s}\cdot F'_{i-1}\right)\cdot\frac{1}{i\cdot s} \quad \text{and} \tag{97}$$

Differentiating one time more:

$$F''_{i+1} = \left(D_t^2 L+(2i-1)(2F'_i+t\cdot F''_i)-\frac{(i-1)}{s}\cdot F''_{i-1}\right)\cdot\frac{1}{i\cdot s} \tag{98}$$

with $D_t L = -\frac{s}{L}$ and $D_t^2 L = -\frac{s^2}{L^3}$.

As in the case where i was less than or equal to zero, we must now compute the first two terms in the recursion formula, i.e.

$$F_1,\ F'_1,\ F''_1,\ F_2,\ F'_2 \text{ and } F''_2$$

Using G. R. 2.2641 we get

$$F_1 = \int \frac{ds}{L} + a_1 = \ln(2\cdot L+2\cdot s-2t)+a_1$$

and hence, by (96)

$$F_2 = \frac{1}{s}\left[\int\frac{s\,ds}{L}+a_2\right] = \frac{1}{s}\left[L+t\int\frac{ds}{L}+a_2\right]$$

Computation of the limites of the integrals for $s\to 0$ give us the integration constants:

$$a_1 = -\ln(2-2t) \quad \text{and}$$

$$a_2 = -t\ln(2-2t)-1.$$

Hence

$$F_1 = \ln\left(\frac{L+s-t}{1-t}\right) = \ln\left(1+\frac{2s}{1-s+L}\right), \tag{99}$$

which can be verified by multiplying the numerator and the denominator by $(1-s+L)$. The last expression for $F_1$ is the best suited for numerical use, because it avoids dividing by zero for $\psi = 0$.

For $F_2$ we get:

$$F_2 = \frac{1}{s}(L+t\cdot F_1+a_2) = \frac{1}{s}(L-1+t\cdot F_1). \tag{100}$$

The first and second derivatives of $F_1$ and $F_2$ becomes:

$$F_1' = \frac{(1-s+L)}{(1+s+L)} \cdot \frac{(-2s)(-s/L)}{(1-s+L)^2} = \frac{2s^2}{(1+s+L)(1-s+L)L} \tag{101}$$

$$= \frac{2s^2}{(1+L-ts)\cdot L\cdot 2} = \frac{s^2}{(1+L-ts)\cdot L} = \frac{s^2}{L\cdot N} .$$

$$F_1'' = s^2 \left[\frac{-N(-s/L) - L(-s/L-s)}{L^2N^2}\right] = s^2 \left[\frac{1+L-ts+L+L^2}{L^3 N^2}\right] \tag{102}$$

$$= s^3 \left[\frac{1+L}{L^2N^2} + \frac{1}{N\cdot L^3}\right]$$

$$F_2' = \frac{1}{s}\,(-s/L + t \cdot F_1' + F_1) = -\frac{1}{L} + \frac{ts}{L\cdot N} + F_1/s \tag{103}$$

$$F_2'' = \frac{1}{s}\,(-s^2/L^3 + 2F_1' + t \cdot F_1'') \tag{104}$$

We will now derive the relations between the functions $F_i$ and the covariance models 2, 3, 4 and 5.

Model 2.

Using (59), (61), (66), we get:

$$\mathrm{cov}_2(T_P, T_Q) = K_2(P, Q) = \sum_{\ell=0}^{\infty} \sigma_{2,\ell}(T, T)\cdot s^{\ell+1} P_\ell(t)$$

$$= \sum_{\ell=2}^{\infty} \frac{R^2}{(\ell-1)^2}\, \sigma_{2,\ell}(\Delta g,\ \Delta g)\, s^{\ell+1} P_\ell(t)$$

$$= A_2 \cdot R^2 \sum_{\ell=2}^{\infty} \frac{1}{\ell\cdot(\ell-1)}\, s^{\ell+1} P_\ell(t)$$

$$= A_2 \cdot R^2 \left(\sum_{\ell=2}^{\infty} \left(\frac{1}{\ell-1} - \frac{1}{\ell}\right) s^{\ell+1} P_\ell(t)\right)$$

and by (73), (84) and (86)

$$\mathrm{cov}_2(T_P, T_Q) = A_2 R^2 [(F_{-1} + ts^2 - F_0] = A_2 R^2 [s(M + ts \cdot \ell n \frac{2}{N} + ts) - s \cdot \ell n \frac{2}{N}] \tag{105}$$

$$= A_2 R^2 \cdot s[1 - L + (ts - 1)\ell n \frac{2}{N}].$$

In the same way, we get using (64), (63), (66) and (84)

$$\mathrm{cov}_2(T_P, \Delta g) = \frac{R}{r'} \sum_{\ell=0}^{\infty} \sigma_{2,\ell} (\Delta g, T) s^{\ell+1} P_\ell(t) = A_2 \cdot R \cdot \left(\frac{R}{r'}\right) \cdot \sum_{\ell=2}^{\infty} \frac{s^{\ell+1}}{\ell} P_\ell(t) \tag{106}$$

$$= A_2 \cdot \frac{R^2}{r} \cdot (F_0 - ts^2) = A_2 \cdot \frac{R^2}{r} \cdot s(\ell n \frac{2}{N} - ts)$$

and by (60), (66), (73), (74) and (84):

$$\mathrm{cov}_2(\Delta g_P, \Delta g_Q) = A_2 \sum_{\ell=2}^{\infty} \frac{\ell - 1}{\ell} s^{\ell+2} P_\ell(t)$$

$$= A_2 \cdot s\left(\sum_{\ell=2}^{\infty} s^{\ell+1} P_\ell(t) - \sum_{\ell=2}^{\infty} \frac{s^{\ell+1}}{\ell} P_\ell(t)\right) \tag{107}$$

$$= A_2 s(F - s - ts^2 - (F_0 - s^2 t)) = A_2 s^2 \left(\frac{1}{L} - 1 - \ell n \frac{2}{N}\right)$$

The covariance functions involving deflections of the vertical will, as mentioned above contain $K_2'$, $K_2''$ and $-D_r K_2' - \frac{2}{r} K_2'$.

Differentiating (105) gives:

$$K_2' = A_2 R^2 (F_{-1}' + s^2 - F_0') \qquad \text{and} \tag{108}$$

$$K_2'' = A_2 R^2 (F_{-1}'' - F_0'') \tag{109}$$

Because $-D_r K_2 - \frac{2}{r} K_2' = \mathrm{cov}_2(\Delta g_P, T_Q)$ we get by differentiating (106):

$$-D_r K_2' - \frac{2}{r} K_2' = A_2 \frac{R^2}{r} (F_0' - s^2) \tag{110}$$

Combining the three last equations with (55), (57), (51), and (53) we get the following equations, which can be evaluated using equations (88) - (91).

$$\begin{aligned} \mathrm{cov}_2(\ell_P, \ell_Q) &= (t \cdot K_2' - \sin^2\psi \cdot K_2'')/(G \cdot G' \cdot r \cdot r') \\ &= A_2 \cdot \frac{R^2}{r \cdot r'} (t(F_{-1}' - s^2 - F_0') - \sin^2\psi \cdot (F_{-1}'' - F_0''))/(G \cdot G') \end{aligned} \tag{111}$$

$$\mathrm{cov}_2(m_P, m_Q) = K_2'/(G \cdot G' \cdot r \cdot r') = A_2 \frac{R^2}{r \cdot r'} (F_{-1}' - s^2 - F_0')/(G \cdot G') \tag{112}$$

$$\mathrm{cov}_2(\ell_P, \zeta_Q) = \sin\psi \cdot K_2'/(G \cdot G' \cdot r) = A_2 \frac{R^2}{r} (F_{-1}' - s^2 - F_0') \cdot \sin\psi/(G \cdot G') \tag{113}$$

$$\mathrm{cov}_2(\ell_P, \Delta g_Q) = A_2 \cdot \frac{R^2}{r \cdot r'} \cdot (F_0' - s^2) \cdot \sin\psi \cdot \frac{1}{G} = \frac{A_2 \cdot s}{G} \cdot \sin\psi (F_0' - s^2) \tag{114}$$

Model 3.

From (59), (61) and (67) we get:

$$\mathrm{cov}_3(T_P, T_Q) = K_3(P,Q) = \sum_{\ell=3}^{\infty} \frac{R^2}{(\ell-1)^2} \cdot \sigma_{3,\ell}(\Delta g, \Delta g) s^{\ell+1} P_\ell(t)$$

$$= A_3 R^2 \sum_{\ell=3}^{\infty} \frac{1}{(\ell-1)(\ell-2)} s^{\ell+1} P_\ell(t) = A_3 R^2 \sum_{\ell=3}^{\infty} \left(\frac{1}{\ell-2} - \frac{1}{\ell-1}\right) s^{\ell+1} P_\ell(t)$$

and then using (73):

$$\mathrm{cov}_3(T_P, T_Q) = A_3 R^2 \cdot [F_{-2} - (F_{-1} - s^3 P_2(t))] \tag{115}$$

For the covariances between the gravity anomaly and the anomalous potential we get using (62), (63), (67) and (73):

$$\mathrm{cov}_3(T_P, \Delta g_Q) = \sum_{\ell=3}^{\infty} \left(\frac{R}{r'}\right) \cdot \frac{R}{\ell-1} \sigma_{3,\ell}(\Delta g, \Delta g)\, s^{\ell+1} P_\ell(t)$$

$$= A_3 \frac{R^2}{r'} \sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+1} P_\ell(t) \tag{116}$$

$$= A_3 \frac{R^2}{r'} \cdot F_{-2} \,.$$

And for $\mathrm{cov}_3(\Delta g_P, \Delta g_Q)$ we get using (60), (67), (71), (73) and (74):

$$\mathrm{cov}_3(\Delta g_P, \Delta g_Q) = A_3 \sum_{\ell=3}^{\infty} \frac{\ell-1}{\ell-2} s^{\ell+2} P_\ell(t) = A_3 \cdot s \left[\sum_{\ell=3}^{\infty} s^{\ell+1} P_\ell(t) + \sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+1} P_\ell(t)\right] \tag{117}$$

$$= A_3 \cdot s \left[\frac{s}{L} - s - s^2 t - s^3 P_2(t) + F_{-2}\right]$$

The formula (115) becomes using (86) and (87):

$$\mathrm{cov}_3(T_P, T_Q) = A_3 R^2 \left[-s(M + ts \cdot \ln \frac{2}{M}) + s^3 P_2(t) + s(M(3ts+1)/2 + s^2(P_2(t) \cdot \ln \frac{2}{N} + (1-t^2)/4))\right] \tag{118}$$

$$= A_3 R^2 \cdot \left[s^3 (P_2(t)(1 + \ln \frac{2}{N}) + \frac{\sin^2\psi}{4}) - s^2 \cdot t \ln \frac{2}{M} + s(3ts-1) \cdot \frac{M}{2}\right]$$

This is the correct version of the formula given by Lauritzen (1973, p. 82), in which the quantities here called M and N have been interchanged and the $R^2$ factor is missing.

Explicit expressions can be written down for (116) and (117) as well, using (86) and (87). But generally it is easier to compute the values of (86) and (87) separately and then evaluate the covariances using (115)-(117).

The derivatives necessary for the evaluation of the covariances involving deflections ((38) - (44) and (51) - (57)) becomes by differentiating (115) and (116):

$$K_3' = A_3 R^2 [ F_{-2}' + 3s^3 t - F_{-1}' ] \tag{119}$$

$$K_3'' = A_3 R^2 [ F_{-2}'' + 3s^3 - F_{-1}'' ] \quad \text{and} \tag{120}$$

$$-D_r K_3' - \frac{2}{r} K_3' = D_t \operatorname{cov}_3 (\Delta g_P, T_Q) = \frac{A_3 \cdot R^2}{r} \cdot F_{-2}', \tag{121}$$

which then can be evaluated using the formula for $F_{-1}'$, $F_{-2}'$, $F_{-1}''$, and $F_{-2}''$, (90) - (93). Combining the three last equations with (55), (57), (51), and (53) we get:

$$\operatorname{cov}_3(\ell_P, \ell_Q) = (t \cdot K_3' - \sin^2\psi \cdot K_3'')/(G \cdot G' \cdot r \cdot r') \tag{122}$$

$$= A_3 \cdot s \cdot [t(F_{-2}' + 3ts^3 - F_{-1}') - \sin^2\psi(F_{-2}'' + 3s^3 - F_{-1}'')] \cdot \frac{1}{G \cdot G'}$$

$$\operatorname{cov}_3(m_P, m_Q) = K_3'/(G \cdot G' \cdot r \cdot r') = A_3 \cdot s(F_{-2}' + 3ts^3 - F_{-1}'), \tag{123}$$

$$\operatorname{cov}_3(\ell_P, \zeta_Q) = \sin\psi \cdot K_3'/(G \cdot G' \cdot r) = A_3 \frac{R^2}{r \cdot G \cdot G'}(F_{-2}' + 3ts^3 - F_{-1}') \cdot \sin\psi \tag{124}$$

$$\operatorname{cov}_3(\ell_P, \Delta g_Q) = \sin\psi(-D_{r'}K' - \frac{2}{r'}K')/(G \cdot r) = A_3 \cdot s \cdot \sin\psi \cdot F_{-2}' \cdot \frac{1}{G} \tag{125}$$

<u>Model 4</u>. Using again (59) and (61) and now (68) we get

$$\operatorname{cov}_4(T_P, T_Q) = K_4(P, Q) = \sum_{\ell=3}^{\infty} \frac{R^2}{(\ell-1)^2} \sigma_{4,\ell}(\Delta g, \Delta g) \cdot s^{\ell+1} P_\ell(t)$$

$$= A_4 \cdot R^2 \cdot \sum_{\ell=3}^{\infty} \cdot \frac{1}{(\ell-1)(\ell-2)(\ell+B)} \cdot s^{\ell+1} \cdot P_\ell(t). \tag{126}$$

Unfortunately we will now have to introduce one more notation related to the degree-variances. We will define:

$$\tau_{k,\ell}(T, T) = \sigma_{k,\ell}(T, T) \cdot \frac{1}{A_k \cdot R^2}, \tag{127}$$

$$\tau_{k,\ell}(\Delta g, T) = \sigma_{k,\ell}(\Delta g, T) \cdot \frac{1}{A_k \cdot R} \qquad \text{and} \tag{128}$$

$$\tau_{k,\ell}(\Delta g, \Delta g) = \sigma_{k,\ell}(\Delta g, \Delta g) \cdot \frac{1}{A_k} \; . \tag{129}$$

All the quantities (127) - (129) are unitless quantities, and we have e.g. using (127), (61) and (68):

$$\tau_{4,\ell}(T,T) = \frac{R^2}{(\ell-1)^2}\,\sigma_{4,\ell}(\Delta g, \Delta g) \cdot \frac{1}{A_4 \cdot R^2} = \frac{1}{(\ell-1)(\ell-2)(\ell+B)}$$

This quantity can be partitioned as follows:

$$\tau_{4,\ell}(T,T) = \frac{1}{\ell+B}\left[\frac{1}{\ell-2} - \frac{1}{\ell-1}\right] = \frac{1}{B+2}\left(\frac{1}{\ell-2} - \frac{1}{\ell+B}\right) - \frac{1}{B+1}\left(\frac{1}{\ell-1} - \frac{1}{\ell+B}\right)$$

$$= \frac{1}{(B+2)(B+1)}\left[\frac{B+1}{\ell-2} - \frac{B+2}{\ell-1} + \frac{1}{\ell+B}\right] .$$

hence using (126), (127), (72) and (73) we get:

$$\begin{aligned} \operatorname{cov}_4(T_P, T_Q) = K_4(P,Q) = \frac{A_4 \cdot R^2}{(B+2)(B+1)} \Bigg[ & \sum_{\ell=3}^{\infty} \frac{B+1}{\ell-2} s^{\ell+1} P_\ell(t) - \sum_{\ell=3}^{\infty} \frac{B+2}{\ell-1} s^{\ell+1} P_\ell(t) \\ & + \sum_{\ell=3}^{\infty} \frac{1}{\ell+B} s^{\ell+1} P_\ell(t) \Bigg] \end{aligned} \tag{130}$$

$$\begin{aligned} = \frac{A_4 \cdot R^2}{(B+2)(B+1)} \Bigg[ & (B+1) \cdot F_{-2} - (B+2)(F_{-1} - s^3 P_2(t)) \\ & + F_B - \frac{s}{B} - \frac{s^2 t}{B+1} - \frac{s^3 P_2(t)}{B+2} \Bigg] \end{aligned}$$

Correspondingly we get using (128), (68) and (63):

$$\tau_{4,\ell}(\Delta g, T) = \frac{1}{(\ell-2)(\ell+B)} = \frac{1}{B+2}\left[\frac{1}{\ell-2} - \frac{1}{\ell+B}\right] \text{ and hence using (64), (72)}$$

and (73):

$$\mathrm{cov}_4(\Delta g_P, T_Q) = A_4 \frac{R^2}{r}\left[\sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+1} P_\ell(t) - \sum_{\ell=3}^{\infty} \frac{1}{\ell+B} s^{\ell+1} P_\ell(t)\right] \cdot \frac{1}{B+2} \tag{131}$$

$$= \frac{A_4 R^2}{r\cdot(B+2)} \left[ F_{-2} - \left(F_B - \frac{s}{B} - \frac{s^2 t}{B+1} - \frac{s^3 P_2(t)}{B+2}\right)\right].$$

For $\tau_{4,\ell}(\Delta g, \Delta g)$ we get in a similar way using (129) and (68):

$$\tau_{4,\ell}(\Delta g, \Delta g) = \frac{\ell-1}{(\ell-2)(\ell+B)} = \frac{\ell-2+1}{(\ell-2)(\ell+B)} = \frac{1}{\ell+B} + \left(\frac{1}{\ell-2} - \frac{1}{\ell+B}\right)\frac{1}{B+2}$$

$$= \frac{B+1}{(B+2)(\ell+B)} + \frac{1}{(B+2)(\ell-2)} = \frac{1}{(B+2)}\left(\frac{B+1}{\ell+B} + \frac{1}{\ell-2}\right)$$

and hence using (60), (72) and (73)

$$\mathrm{cov}_4(\Delta g_P, \Delta g_Q) = \frac{A_4}{(B+2)}\left(\sum_{\ell=3}^{\infty} \frac{B+1}{\ell+B} s^{\ell+2} P_\ell(t) + \sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+2} P_\ell(t)\right) \tag{132}$$

$$= \frac{A_4 \cdot s}{(B+2)} \left[(B+1)\left(F_B - \frac{s}{B} - \frac{s^2 t}{B+1} - \frac{s^3 P_2(t)}{B+2}\right) + F_{-2}\right]$$

We will now differentiate (130) and (131) getting the formula necessary for the computation of the covariances involving deflections;

$$K_4' = \frac{A_4 \cdot R^2}{(B+2)(B+1)}\left[(B+1)F_{-2}' - (B+2)(F_{-1}' - 3ts^3) + F_B' - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2}\right] \tag{133}$$

$$K_4'' = \frac{A_4 \cdot R^2}{(B+2)(B+1)}\left[(B+1)F_{-2}'' - (B+2)(F_{-1}'' - 3s^3) + F_B'' - \frac{3s^3}{B+2}\right] \tag{134}$$

$$-D_r K' - \frac{2}{r} K_4' = D_t(\mathrm{cov}(\Delta g_P, T_Q)) = \frac{A_4 R^2}{r\cdot(B+2)}\left[F_{-2}' - \left(F_B' - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2}\right)\right] \tag{135}$$

The formula (133)-(135) can be evaluated using (90)-(93) and the recursion formula (97) and (98) with the "initial values" given by (101)-(104).

By using (133)-(135) we can write down the covariance functions (55), (57), (51), and (53). We get:

$$\mathrm{cov}_4(\ell_P, \ell_Q) = (t\cdot K_4' - \sin^2\psi K_4'')/(G \cdot G' \cdot r \cdot r') \tag{136}$$

$$= \frac{A_4 \cdot s}{(B+2)(B+1)} \left[ t\cdot \left( (B+1) F_{-2}' - (B+2)(F_{-1}' - 3ts^3) + F_B' - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2} \right) \right.$$

$$\left. \div \sin^2\psi \left( (B+1) F_{-2}'' - (B+2)(F_{-1}'' - 3s^3) + F_B'' - \frac{3s^3}{B+2} \right) \right] \cdot \frac{1}{G\cdot G'},$$

$$\mathrm{cov}_4(m_P, m_Q) = K_4'/(G\cdot G' \cdot r \cdot r') \tag{137}$$

$$= \frac{A_4 \cdot s}{(B+2)(B+1)\cdot G\cdot G'} \left[ (B+1) F_{-2}' - (B+2)(F_{-1}' - 3ts^3) + F_B' - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2} \right]$$

$$\mathrm{cov}_4(\ell_P, \zeta_Q) = \sin\psi \cdot K_4'/(G\cdot G' \cdot r) \tag{138}$$

$$= \frac{A_4 \cdot R^2}{(B+2)(B+1) r\cdot G\cdot G'} \cdot \left[ (B+1) F_{-2}' - (B+2)(F_{-1}' - 3ts^3) + F_B' \right.$$

$$\left. - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2} \right]$$

and finally:

$$\mathrm{cov}_4(\ell_P, \Delta g_Q) = \sin\psi \cdot (-D_{r'} K_4' - \frac{2}{r} K_4') \cdot \frac{1}{G\cdot r} \tag{139}$$

$$= \frac{A_4 \cdot \sin\psi \cdot s}{(B+2)\cdot G} \left[ F_{-2}' - (F_B' - \frac{s^2}{B+1} - \frac{3s^3 t}{B+2}) \right]$$

<u>Model 5.</u> Using (127), (61) and (69) we get:

$$\tau_{5,\ell}(T, T) = \frac{R^2}{(\ell-1)} \sigma_{5,\ell}(\Delta g, \Delta g) \frac{1}{A_5 \cdot R^2} = \frac{1}{(\ell-1)(\ell-2)(\ell+i)(\ell+j)}$$

$$= \frac{1}{j-i} \left[ \frac{1}{(\ell-1)(\ell-2)(\ell+i)} - \frac{1}{(\ell-1)(\ell-2)(\ell+j)} \right]$$

$$= \frac{1}{j-i}\left[\left(\frac{1}{(\ell-2)(i+2)} - \frac{1}{(\ell-1)(i+1)} + \frac{1}{(\ell+i)(i+1)(i+2)}\right)\right.$$

$$\left. - \left(\frac{1}{(\ell-2)(j+2)} - \frac{1}{(\ell-1)(j+1)} + \frac{1}{(\ell+j)(j+1)(j+2)}\right)\right]$$

$$= \frac{1}{j-i}\left[\frac{j+2-i-2}{(\ell-2)(i+2)(j+2)} + \frac{i+1-j-1}{(\ell-1)(j+1)(i+1)} + \frac{1}{(\ell+i)(\ell+1)(i+2)}\right.$$

$$\left. - \frac{1}{(\ell+j)(j+1)(j+2)}\right]$$

$$= \frac{1}{(\ell-2)(i+2)(j+2)} - \frac{1}{(\ell-1)(i+1)(j+1)} + \left[\frac{1}{(\ell+i)(i+1)(i+2)}\right.$$

$$\left. - \frac{1}{(\ell+j)(j+1)(j+2)}\right]\frac{1}{j-i} ,$$

and by (128), (63) and (69)

$$\tau_{5,\ell}(\Delta g, T) = \frac{1}{(\ell-2)(\ell+i)(\ell+j)} = \frac{1}{j-i}\left[\frac{1}{\ell-2}\cdot\frac{1}{\ell+i} - \frac{1}{\ell-2}\cdot\frac{1}{\ell+j}\right]$$

$$= \frac{1}{j-i}\left[\frac{1}{i+2}\left(\frac{1}{\ell-2} - \frac{1}{\ell+i}\right) - \frac{1}{j+2}\left(\frac{1}{\ell-2} - \frac{1}{\ell+j}\right)\right]$$

$$= \frac{1}{\ell-2} + \frac{1}{j-i}\left[\frac{1}{(j+2)(\ell+j)} - \frac{1}{(\ell+i)(i+2)}\right]$$

and finally by (129) and (69)

$$\tau_{5,\ell}(\Delta g, \Delta g) = \frac{\ell-1}{(\ell-2)(\ell+i)(\ell+j)} = \frac{1}{j-i}\left[\frac{i+1}{i+2}\cdot\frac{1}{\ell+i} + \frac{1}{(\ell-2)(i+2)}\right.$$

$$\left. - \frac{j+1}{j+2}\cdot\frac{1}{\ell+j} - \frac{1}{(\ell-2)(j+2)}\right]$$

$$= \frac{1}{(\ell-2)(i+2)(j+2)} + \frac{1}{j-i}\left[\frac{i+1}{i+2}\cdot\frac{1}{\ell+i} - \frac{j+1}{j+2}\cdot\frac{1}{\ell+j}\right]$$

and hence using (59), (72) and (73) we get:

$$\begin{aligned} \mathrm{cov}_5(T_P, T_Q) = K_5(P, Q) = A_5 \cdot R^2 \Bigg[ & \frac{1}{(i+2)(j+2)} \sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+1} P_\ell(t) \\ & - \frac{1}{(i+1)(j+1)} \sum_{\ell=3}^{\infty} \frac{1}{\ell-1} s^{\ell+1} P_\ell(t) \\ & + \frac{1}{j-i}\left(\frac{1}{(i+1)(i+2)} \sum_{\ell=3}^{\infty} \frac{1}{\ell+i} s^{\ell+1} P_\ell(t) \right. \\ & \left. - \frac{1}{(j+1)(j+2)} \sum_{\ell=3}^{\infty} \frac{1}{\ell+j} s^{\ell+1} P_\ell(t)\right)\Bigg] \end{aligned} \tag{140}$$

$$\begin{aligned} = A_5 R^2 \Bigg[ & \frac{1}{(i+2)(j+2)} F_{-2} - \frac{1}{(i+1)(j+1)} (F_{-1} - s^3 P_2(t)) \\ & + \frac{1}{j-i}\left(\frac{1}{(i+1)(i+2)}\left(F_i - \frac{s}{i} - \frac{s^2 t}{i+1} \quad \frac{s^3 P_2(t)}{i+2}\right) - \frac{1}{(j+1)(j+2)}\left(F_j - \frac{s}{j}\right.\right. \\ & \left.\left. - \frac{s^2 t}{j+1} - \frac{s^3 P_2(t)}{j+2}\right)\right)\Bigg], \end{aligned}$$

by(62)

$$\begin{aligned} \mathrm{cov}_5(\Delta g_P, T_Q) = A_5 \frac{R^2}{r}\Bigg[ & \sum_{\ell=3}^{\infty} \frac{1}{\ell-2} s^{\ell+1} P_\ell(t) + \frac{1}{j-i}\left(\frac{1}{j+2} \sum_{\ell=3}^{\infty} \frac{1}{\ell+j} s^{\ell+1} P_\ell(t)\right. \\ & \left. - \frac{1}{i+2} \sum_{\ell=3}^{\infty} \frac{1}{\ell+i} s^{\ell+1} P_\ell(t)\right)\Bigg] \end{aligned} \tag{141}$$

$$=A_5 \frac{R^2}{r}\left[F_{-2} + \frac{1}{j-i}\left(\frac{1}{j+2}\left(F_j - \frac{s}{j} - \frac{s^2 t}{j+1} - \frac{s^2 P_2(t)}{j+2}\right)\right.\right.$$

$$\left.\left. - \frac{1}{i+2}\left(F_i - \frac{s}{i} - \frac{s^2 t}{i+1} - \frac{s^3 P_2(t)}{i+2}\right)\right)\right] \qquad (141) \text{ cont'd}$$

and by (60)

$$\mathrm{cov}_5(\Delta g_P, \Delta g_Q) = A_5\left[\frac{1}{(i+2)(j+2)}\sum_{\ell=3}^{\infty}\frac{1}{\ell-2}s^{\ell+2}P_\ell(t)\right.$$

$$\left. + \frac{1}{j-i}\left(\frac{i+1}{i+2}\sum_{\ell=3}^{\infty}\frac{1}{\ell+i}s^{\ell+2}P_\ell(t) - \frac{j+1}{j+2}\sum_{\ell=3}^{\infty}\frac{1}{\ell+j}s^{\ell+2}P_\ell(t)\right)\right] \qquad (142)$$

$$= A_5 \cdot s\left[\frac{1}{(i+2)(j+2)}F_{-2} + \frac{1}{j-i}\left(\frac{i+1}{i+2}\left(F_i - \frac{s}{i} - \frac{s^2}{i+1} - \frac{s^3 P_2(t)}{i+2}\right)\right.\right.$$

$$\left.\left. - \frac{j+1}{j+2}\left(F_j - \frac{s}{j} - \frac{s^2 t}{j+1} - \frac{s^3 P_2(t)}{j+2}\right)\right)\right].$$

The covariances (140)-(142) can then be evaluated using (86), (87) and the recursion formula (96) with "initial values" (99) and (100). As in the other models it is necessary to compute $K_5'$, $K_5''$ and $-D_r K_5' - \frac{2}{r}K_5'$ to find the expressions for the covariance functions involving deflections of the vertical. The formulae can be derived by differentiating (140) and (141) and later evaluated using the proper recursion formula exactly as explained in model 4.

Note in the equations (140), (141) and (142) the denominators are equal to j-i, i+2, j+2, i+1, j+1. The occurence of these and similar quantities are the reason for the above mentioned restrictions on i and j (and B).

The above described expressions for the closed covariance functions can also be used in cases, where a set of empirical degree-variances are used in connection with degree-variances defined through one of the models (65)-(69). In this case, the basic covariance function cov $(T_P, T_Q)$ is represented by, e.g.

$$\sum_{\ell=0}^{n} \hat{\sigma}_{\ell}(T,T)s^{\ell+1} P_{\ell}(t) + \sum_{\ell=n+1}^{\infty} \sigma_{k,\ell}(T,T)s^{\ell+1}P_{\ell}(t) \tag{143}$$

where $\hat{\sigma}_{\ell}(T,T)$ are the empirically determined degree-variances as would be computed from equation (15). We will distinguish between the above mentioned covariance functions $cov_k(A,B)$ and this new type of covariance function by a subscript E, i.e., $cov_E(T_P, T_Q)$ is equal to expression (143). We rewrite (143):

$$cov_E(T_P, T_Q) = \sum_{\ell=0}^{n} (\hat{\sigma}_{\ell}(T,T) - \sigma_{k,\ell}(T,T))s^{\ell+1}P_{\ell}(t) + \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T,T)s^{\ell+1}P_{\ell}(t)$$

$$= \sum_{\ell=0}^{n} (\hat{\sigma}_{\ell}(T,T) - \sigma_{k,\ell}(T,T))s^{\ell+1}P_{\ell}(t) + cov_k(T_P, T_Q). \tag{144}$$

Noting, that the relations (61) and (63) are valid for empirical degree-variances as well, we find using (60), (63), and the relations (34) - (37), (51), (53), (55) and (57)

$$cov_E(\Delta g_P, T_Q) = \sum_{\ell=0}^{n} (\hat{\sigma}_{\ell}(\Delta g, T) - \sigma_{k,\ell}(\Delta g, T)) \cdot \frac{R}{r} s^{\ell+1} P_{\ell}(t) + cov_k(\Delta g_P, T_Q), \tag{145}$$

$$cov_E(\Delta g_P, \Delta g_Q) = \sum_{\ell=0}^{n} (\hat{\sigma}_{\ell}(\Delta g, \Delta g) - \sigma_{k,\ell}(\Delta g, \Delta g))s^{\ell+2} P_{\ell}(t) + cov_k(\Delta g_P, \Delta g_Q) \tag{146}$$

$$cov_E(\ell_P, \ell_Q) = \Big(\sum_{\ell=0}^{n} (\hat{\sigma}_{\ell}(T,T) - \sigma_{k,\ell}(T,T))\, s^{\ell+1}(t \cdot P'_{\ell}(t) - \sin^2\psi \cdot P''_{\ell}(t)\Big)/(G \cdot G' \cdot r \cdot r')$$

$$+ cov_k(\ell_P, \ell_Q) \tag{147}$$

$$= \Big(t \cdot \sum_{\ell=0}^{n} (\hat{\sigma}_{\ell}(T,T) - \sigma_{k,\ell}(T,T))s^{\ell+1}P'_{\ell}(t) - \sin^2\psi \sum_{\ell=0}^{n} (\hat{\sigma}_{\ell}(T,T) - \sigma_{k,\ell}(T,T))$$

$$\cdot s^{\ell+1}P''_{\ell}(t)\Big)/(G \cdot G' \cdot r \cdot r') + cov_k(\ell_P, \ell_Q)$$

$$\mathrm{cov}_E(m_P, m_Q) = \Big(\sum_{\ell=0}^{n} (\hat{\sigma}_\ell(T,T) - \sigma_{k,\ell}(T,T)) s^{\ell+1} P'_\ell(t)\Big) / (G \cdot G' \cdot r \cdot r') \tag{148}$$

$$+\mathrm{cov}_k(m_P, m_Q)$$

$$\mathrm{cov}_E(\ell_P, \ell_Q) = \sin\psi\Big(\sum_{\ell=0}^{n} (\hat{\sigma}_\ell(T,T) - \sigma_{k,\ell}(T,T)) s^{\ell+1} P_\ell(t)\Big) / (G \cdot G' \cdot r) + \mathrm{cov}(\ell_P, \ell_Q) \tag{149}$$

and finally

$$\mathrm{cov}_E(\ell_P, \Delta g_Q) = \sin\psi\Big(\sum_{\ell=0}^{n} (\hat{\sigma}_\ell(\Delta g, T) - \sigma_{k,\ell}(\Delta g, T)) s^{\ell+1} \cdot \frac{R}{r'} \cdot P'_\ell(t)\Big) / (G \cdot r) \tag{150}$$

$$+\mathrm{cov}_k(\ell_P, \Delta g_Q)$$

where $P'_\ell(t)$ and $P''_\ell(t)$ are the $\ell$'th order Legendre polynomial differentiated with respect to t one and two times respectively.

We now define $\epsilon_\ell$ through the following equations:

$$\epsilon_\ell(T,T) = \hat{\sigma}_\ell(T,T) - \sigma_{k,\ell}(T,T) \tag{151}$$

$$\epsilon_\ell(\Delta g, T) = \hat{\sigma}_\ell(\Delta g, T) - \sigma_{k,\ell}(\Delta g, T) \text{ and} \tag{152}$$

$$\epsilon_\ell(\Delta g, \Delta g) = \hat{\sigma}_\ell(\Delta g, \Delta g) - \sigma_{k,\ell}(\Delta g, \Delta g) \tag{153}$$

We can then see, that the covariance functions (144)-(150) involves the summation of finite series.

$$\sum_{\ell=0}^{n} \epsilon_\ell(T,T) s^{\ell+1} P_\ell(t), \sum_{\ell=0}^{n} \epsilon_\ell(T,\Delta g) s^{\ell+1} P_\ell(t), \sum_{\ell=0}^{n} \epsilon_\ell(\Delta g, \Delta g) s^{\ell+1} P_\ell(t) \tag{154}$$

$$\sum_{\ell=0}^{n} \epsilon_\ell(T,T) s^{\ell+1} P'_\ell(t), \sum_{\ell=0}^{n} \epsilon_\ell(T,\Delta g) s^{\ell+1} P'_\ell(t) \tag{155}$$

and

$$\sum_{\ell=0}^{n} \epsilon_\ell(T,T)s^{\ell+1}P''_\ell(t) \tag{156}$$

Recursion algorithms for the summation of those three types of series will be given in the next section.

Using the above developed expressions (144)-(150) it is possible to compute covariance functions of and between height anomalies, gravity anomalies and deflections of the vertical corresponding to the recommended model for the anomaly degree-variances. This is possible because we have selected the value B in Table Seven (p. 22 ) equal to the integer 24.

Using the empirical determined value of $\hat{\sigma}_2(\Delta g, \Delta g) = 7.5$ mgal$^2$ (cf. Table Two) we can then, for example, write down an expression for the covariance functions of the anomalous potential:

$$\text{cov}_E(T_P, T_Q) = 7.5 \cdot 10^{-10} \cdot R^2 \cdot s^3 \cdot P_2(t) + A \cdot 10^{-10} \cdot R^2 \sum_{\ell=3}^{\infty} \frac{1}{(\ell-1)(\ell-2)(\ell+24)} s^{\ell+1} P_\ell(t)$$

$$= 7.5 \cdot 10^{-10} \cdot R^2 \cdot s^2 \cdot P_2(t) + \text{cov}_4(T_P, T_P),$$

where the factor $10^{-10}$ is used to convert the covariance into units of m$^4$/sec$^4$, supposing R in units of meters.

In a similar way we can write down the expressions for the covariance functions, $\text{cov}_E(\Delta g_P, \zeta_Q)$, $\text{cov}_E(\Delta g_P, \Delta g_Q)$, $\text{cov}_E(\Delta g_P, \ell_Q)$, $\text{cov}_E(\zeta_P, \ell_Q)$, $\text{cov}_E(\ell_P, \ell_Q)$ and $\text{cov}_E(m_P, m_Q)$.

We have computed values of the covariances for varying spherical distance $\psi$ and for P and Q lying on the surface of the Earth and 500 km above the surface of the Earth respectively. See tables 9 and 10 and figures 3-9.

The radius of the Bjerhammar sphere, R, has been determined as:

$$R = \sqrt{s_{\text{table 7}}} \cdot R_e = (0.999617)^{\frac{1}{2}} \cdot 6371.0\,\text{km} = 6369.8\,\text{km}.$$

The quantities r and r$'$ are computed by adding the actual height above the reference ellipsoid (here 0 and 500 km) to the adopted mean Earth radius, $R_e$.

The subroutine presented in the appendix has been used for the computation of the given values.

Table 9

Covariance between various quantities computed using the anomaly degree variances of model 4 and $\sigma_8(\Delta g, \Delta g) = 7.5 \text{mgal}^2$ at the surface of the sphere approximating the earth ($R_e = 6371$km).

Covariances Between

| $\psi$ | $\Delta g_P, \Delta g_Q$ mgal$^2$ | $\Delta g_P, \ell_Q$ mgal·arc sec | $\Delta g_P, \zeta_Q$ mgal·m | $\ell_P, \ell_Q$ arc sec$^2$ | $m_P, m_Q$ arc sec$^2$ | $\ell_P, \zeta_Q$ arc sec·m | $\zeta_P, \zeta_Q$ m$^2$ |
|---|---|---|---|---|---|---|---|
| 0° 0.0' | 1795.0 | 0.0 | 452.3 | 45.3 | 45.3 | 0.0 | 926.1 |
| 0 30.0 | 801.8 | 67.3 | 434.8 | 19.2 | 27.1 | 7.3 | 925.0 |
| 1 0.0 | 572.7 | 59.9 | 417.7 | 14.1 | 21.7 | 11.7 | 922.4 |
| 1 30.0 | 452.6 | 54.2 | 402.3 | 11.5 | 18.7 | 15.1 | 918.8 |
| 2 0.0 | 375.5 | 49.8 | 388.3 | 9.8 | 16.7 | 18.0 | 914.3 |
| 2 30.0 | 320.9 | 46.2 | 375.4 | 8.6 | 15.2 | 20.4 | 909.1 |
| 3 0.0 | 279.9 | 43.3 | 363.3 | 7.7 | 14.0 | 22.6 | 903.3 |
| 3 30.0 | 247.6 | 40.9 | 352.0 | 7.0 | 13.1 | 24.6 | 896.9 |
| 4 0.0 | 221.6 | 38.8 | 341.3 | 6.4 | 12.3 | 26.4 | 890.0 |
| 5 0.0 | 181.9 | 35.4 | 321.3 | 5.5 | 11.0 | 29.6 | 874.9 |
| 6 0.0 | 152.8 | 32.8 | 303.0 | 4.8 | 10.0 | 32.4 | 858.2 |
| 8 0.0 | 112.8 | 28.9 | 269.8 | 3.7 | 8.6 | 36.9 | 820.7 |
| 10 0.0 | 86.1 | 26.2 | 240.2 | 2.9 | 7.5 | 40.5 | 778.9 |
| 12 0.0 | 66.9 | 24.0 | 213.2 | 2.3 | 6.7 | 43.3 | 733.7 |
| 14 0.0 | 52.2 | 22.3 | 188.3 | 1.7 | 6.1 | 45.4 | 685.9 |
| 16 0.0 | 40.6 | 20.8 | 165.1 | 1.2 | 5.5 | 47.0 | 636.0 |
| 18 0.0 | 31.1 | 19.4 | 143.4 | 0.7 | 5.0 | 48.0 | 584.8 |
| 20 0.0 | 23.2 | 18.2 | 123.2 | 0.3 | 4.6 | 48.6 | 532.7 |
| 22 0.0 | 16.5 | 17.0 | 104.2 | -0.0 | 4.2 | 48.7 | 480.2 |
| 24 0.0 | 10.9 | 15.9 | 86.5 | -0.4 | 3.9 | 48.5 | 427.7 |
| 26 0.0 | 6.0 | 14.8 | 69.9 | -0.7 | 3.5 | 47.9 | 375.7 |
| 28 0.0 | 1.8 | 13.8 | 54.5 | -1.0 | 3.2 | 47.0 | 324.5 |
| 30 0.0 | -1.7 | 12.7 | 40.2 | -1.2 | 3.0 | 45.8 | 274.4 |
| 35 0.0 | -8.6 | 10.3 | 9.2 | -1.8 | 2.4 | 41.7 | 156.2 |
| 40 0.0 | -12.9 | 7.9 | -15.2 | -2.2 | 1.8 | 36.3 | 50.9 |
| 45 0.0 | -15.4 | 5.6 | -33.3 | -2.4 | 1.4 | 30.1 | -38.7 |
| 50 0.0 | -16.3 | 3.5 | -45.6 | -2.5 | 1.0 | 23.4 | -110.8 |
| 55 0.0 | -15.9 | 1.7 | -52.6 | -2.5 | 0.7 | 16.6 | -164.7 |
| 60 0.0 | -14.5 | 0.0 | -54.8 | -2.4 | 0.4 | 10.0 | -200.5 |
| 65 0.0 | -12.4 | -1.3 | -53.0 | -2.2 | 0.1 | 3.9 | -219.2 |
| 70 0.0 | -9.8 | -2.4 | -47.9 | -1.9 | -0.1 | -1.5 | -222.2 |
| 75 0.0 | -6.9 | -3.2 | -40.3 | -1.5 | -0.2 | -6.1 | -211.8 |
| 80 0.0 | -4.0 | -3.6 | -31.0 | -1.2 | -0.3 | -9.7 | -190.3 |
| 85 0.0 | -1.1 | -3.8 | -20.9 | -0.8 | -0.4 | -12.3 | -160.4 |
| 90 0.0 | 1.6 | -3.8 | -10.6 | -0.4 | -0.4 | -13.9 | -124.9 |
| 95 0.0 | 3.9 | -3.5 | -0.8 | -0.1 | -0.5 | -14.5 | -86.5 |
| 100 0.0 | 5.7 | -3.0 | 7.9 | 0.2 | -0.5 | -14.3 | -47.5 |
| 105 0.0 | 6.9 | -2.4 | 15.2 | 0.5 | -0.4 | -13.3 | -10.2 |
| 110 0.0 | 7.6 | -1.7 | 20.7 | 0.7 | -0.4 | -11.7 | 23.7 |
| 115 0.0 | 7.7 | -0.9 | 24.1 | 0.8 | -0.3 | -9.7 | 52.7 |
| 120 0.0 | 7.2 | -0.1 | 25.5 | 0.9 | -0.3 | -7.5 | 76.0 |
| 125 0.0 | 6.2 | 0.6 | 24.9 | 0.9 | -0.2 | -5.2 | 93.0 |
| 130 0.0 | 4.7 | 1.2 | 22.5 | 0.8 | -0.1 | -2.9 | 103.9 |
| 135 0.0 | 2.9 | 1.7 | 18.5 | 0.7 | -0.0 | -0.9 | 108.9 |
| 140 0.0 | 0.8 | 2.1 | 13.4 | 0.6 | 0.0 | 0.9 | 108.9 |
| 145 0.0 | -1.4 | 2.3 | 7.4 | 0.4 | 0.1 | 2.2 | 104.7 |
| 150 0.0 | -3.7 | 2.4 | 1.1 | 0.2 | 0.2 | 3.0 | 97.5 |
| 155 0.0 | -5.8 | 2.3 | -5.2 | 0.1 | 0.3 | 3.4 | 88.7 |
| 160 0.0 | -7.7 | 2.0 | -11.0 | -0.1 | 0.3 | 3.4 | 79.3 |
| 165 0.0 | -9.3 | 1.6 | -15.9 | -0.2 | 0.4 | 2.9 | 70.7 |
| 170 0.0 | -10.5 | 1.1 | -19.7 | -0.3 | 0.4 | 2.1 | 63.8 |
| 175 0.0 | -11.3 | 0.6 | -22.1 | -0.4 | 0.4 | 1.1 | 59.4 |
| 180 0.0 | -11.5 | 0.0 | -22.9 | -0.4 | 0.4 | 0.0 | 57.8 |

Table 10

Covariances between various quantities computed using the anomaly degree variances of model 4 and $\hat{\sigma}_2(\Delta g, \Delta g) = 7.5\ \text{mgal}^2$ at a height 500 km above the earth ($r = R_e + 500\ \text{km}$).

Covariances Between

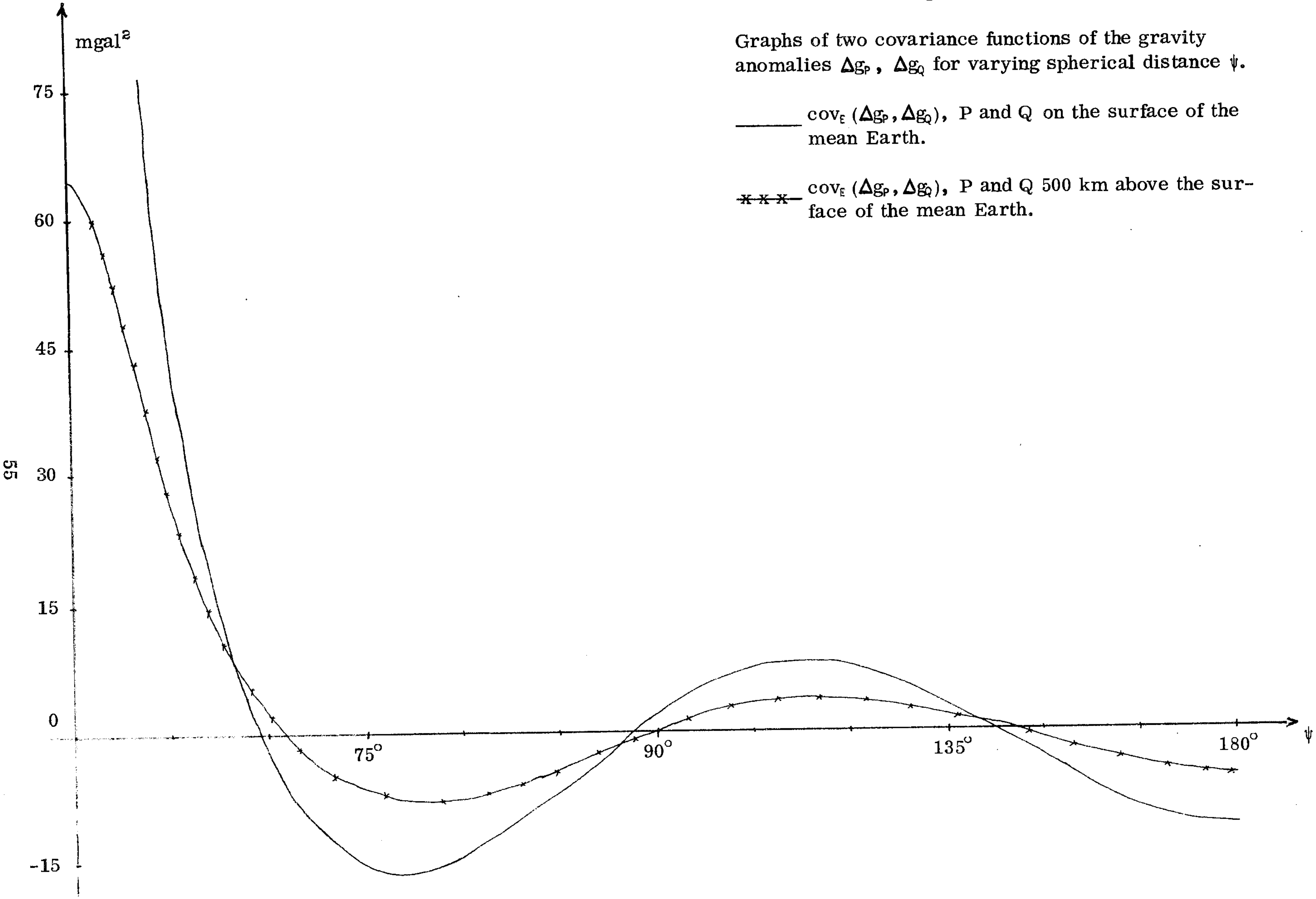
| $\psi$ | $\Delta g_P, \Delta g_Q$ mgal$^2$ | $\Delta g_P, \ell_Q$ mgal·arc sec | $\Delta g_P, \zeta_Q$ mgal·m | $\ell_P, \ell_Q$ arc sec$^2$ | $m_P, m_Q$ arc sec$^2$ | $\ell_P, \zeta_Q$ arc sec·m | $\zeta_P, \zeta_Q$ m$^2$ |
|---|---|---|---|---|---|---|---|
| 0° 0.0' | 64.1 | 0.0 | 162.2 | 4.3 | 4.3 | 0.0 | 648.4 |
| 0 30.0 | 64.0 | 0.6 | 162.1 | 4.3 | 4.3 | 1.2 | 648.3 |
| 1 0.0 | 63.7 | 1.3 | 161.8 | 4.3 | 4.3 | 2.5 | 647.7 |
| 1 30.0 | 63.2 | 1.9 | 161.4 | 4.2 | 4.3 | 3.7 | 646.8 |
| 2 0.0 | 62.5 | 2.5 | 160.7 | 4.2 | 4.3 | 4.9 | 645.5 |
| 2 30.0 | 61.7 | 3.1 | 159.9 | 4.1 | 4.2 | 6.2 | 643.9 |
| 3 0.0 | 60.7 | 3.7 | 158.9 | 4.1 | 4.2 | 7.4 | 642.0 |
| 3 30.0 | 59.5 | 4.2 | 157.7 | 4.0 | 4.2 | 8.5 | 639.7 |
| 4 0.0 | 58.3 | 4.7 | 156.4 | 3.9 | 4.2 | 9.7 | 637.0 |
| 5 0.0 | 55.5 | 5.7 | 153.4 | 3.7 | 4.1 | 11.9 | 630.7 |
| 6 0.0 | 52.4 | 6.5 | 149.8 | 3.6 | 4.0 | 14.0 | 623.2 |
| 8 0.0 | 46.0 | 7.7 | 141.5 | 3.1 | 3.9 | 17.9 | 604.5 |
| 10 0.0 | 39.7 | 8.5 | 132.1 | 2.7 | 3.7 | 21.3 | 581.7 |
| 12 0.0 | 33.7 | 8.9 | 121.9 | 2.3 | 3.5 | 24.2 | 555.2 |
| 14 0.0 | 28.3 | 9.2 | 111.3 | 1.9 | 3.3 | 26.6 | 525.5 |
| 16 0.0 | 23.5 | 9.2 | 100.5 | 1.5 | 3.1 | 28.6 | 493.4 |
| 18 0.0 | 19.1 | 9.1 | 89.8 | 1.2 | 2.9 | 30.2 | 459.2 |
| 20 0.0 | 15.3 | 8.9 | 79.2 | 0.8 | 2.8 | 31.3 | 423.4 |
| 22 0.0 | 11.9 | 8.7 | 68.9 | 0.5 | 2.6 | 32.1 | 386.4 |
| 24 0.0 | 8.9 | 8.4 | 58.8 | 0.3 | 2.4 | 32.6 | 348.8 |
| 26 0.0 | 6.3 | 8.0 | 49.2 | 0.0 | 2.2 | 32.8 | 310.7 |
| 28 0.0 | 3.9 | 7.6 | 40.1 | -0.2 | 2.1 | 32.6 | 272.7 |
| 30 0.0 | 1.9 | 7.2 | 31.4 | -0.4 | 1.9 | 32.2 | 235.0 |
| 35 0.0 | -2.2 | 6.0 | 11.9 | -0.9 | 1.6 | 30.2 | 143.9 |
| 40 0.0 | -5.0 | 4.8 | -4.1 | -1.2 | 1.3 | 27.1 | 60.4 |
| 45 0.0 | -6.7 | 3.5 | -16.5 | -1.5 | 1.0 | 23.1 | -12.6 |
| 50 0.0 | -7.6 | 2.3 | -25.4 | -1.6 | 0.7 | 18.5 | -73.2 |
| 55 0.0 | -7.8 | 1.3 | -31.0 | -1.7 | 0.5 | 13.8 | -120.2 |
| 60 0.0 | -7.3 | 0.3 | -33.5 | -1.6 | 0.3 | 9.0 | -153.3 |
| 65 0.0 | -6.5 | -0.5 | -33.4 | -1.5 | 0.1 | 4.5 | -172.8 |
| 70 0.0 | -5.3 | -1.2 | -31.0 | -1.3 | 0.0 | 0.4 | -179.7 |
| 75 0.0 | -3.9 | -1.7 | -27.0 | -1.1 | -0.1 | -3.2 | -175.4 |
| 80 0.0 | -2.5 | -2.0 | -21.7 | -0.9 | -0.2 | -6.1 | -161.7 |
| 85 0.0 | -1.0 | -2.2 | -15.6 | -0.6 | -0.3 | -8.3 | -140.5 |
| 90 0.0 | 0.3 | -2.2 | -9.3 | -0.4 | -0.3 | -9.8 | -113.9 |
| 95 0.0 | 1.5 | -2.0 | -3.1 | -0.2 | -0.3 | -10.6 | -84.1 |
| 100 0.0 | 2.5 | -1.8 | 2.6 | 0.1 | -0.3 | -10.8 | -52.9 |
| 105 0.0 | 3.2 | -1.4 | 7.5 | 0.2 | -0.3 | -10.3 | -22.1 |
| 110 0.0 | 3.6 | -1.0 | 11.3 | 0.4 | -0.3 | -9.4 | 6.7 |
| 115 0.0 | 3.7 | -0.6 | 13.9 | 0.5 | -0.3 | -8.2 | 32.4 |
| 120 0.0 | 3.6 | -0.2 | 15.3 | 0.5 | -0.2 | -6.7 | 54.1 |
| 125 0.0 | 3.1 | 0.3 | 15.5 | 0.6 | -0.2 | -5.1 | 71.3 |
| 130 0.0 | 2.5 | 0.6 | 14.6 | 0.5 | -0.1 | -3.5 | 83.9 |
| 135 0.0 | 1.6 | 0.9 | 12.7 | 0.5 | -0.1 | -2.1 | 92.1 |
| 140 0.0 | 0.6 | 1.2 | 10.0 | 0.4 | -0.0 | -0.8 | 96.1 |
| 145 0.0 | -0.4 | 1.3 | 6.7 | 0.3 | 0.0 | 0.3 | 96.8 |
| 150 0.0 | -1.5 | 1.3 | 3.2 | 0.2 | 0.1 | 1.0 | 94.9 |
| 155 0.0 | -2.5 | 1.3 | -0.4 | 0.1 | 0.1 | 1.4 | 91.3 |
| 160 0.0 | -3.5 | 1.1 | -3.7 | -0.0 | 0.1 | 1.6 | 86.8 |
| 165 0.0 | -4.2 | 0.9 | -6.5 | -0.1 | 0.2 | 1.4 | 82.4 |
| 170 0.0 | -4.8 | 0.7 | -8.7 | -0.2 | 0.2 | 1.1 | 78.6 |
| 175 0.0 | -5.2 | 0.3 | -10.0 | -0.2 | 0.2 | 0.6 | 76.1 |
| 180 0.0 | -5.3 | 0.0 | -10.5 | -0.2 | 0.2 | 0.0 | 75.3 |

Figure 3

Graphs of two covariance functions of the gravity anomalies $\Delta g_P$, $\Delta g_Q$ for varying spherical distance $\psi$.

———— $\mathrm{cov}_E(\Delta g_P, \Delta g_Q)$, P and Q on the surface of the mean Earth.

-x-x-x- $\mathrm{cov}_E(\Delta g_P, \Delta g_Q)$, P and Q 500 km above the surface of the mean Earth.



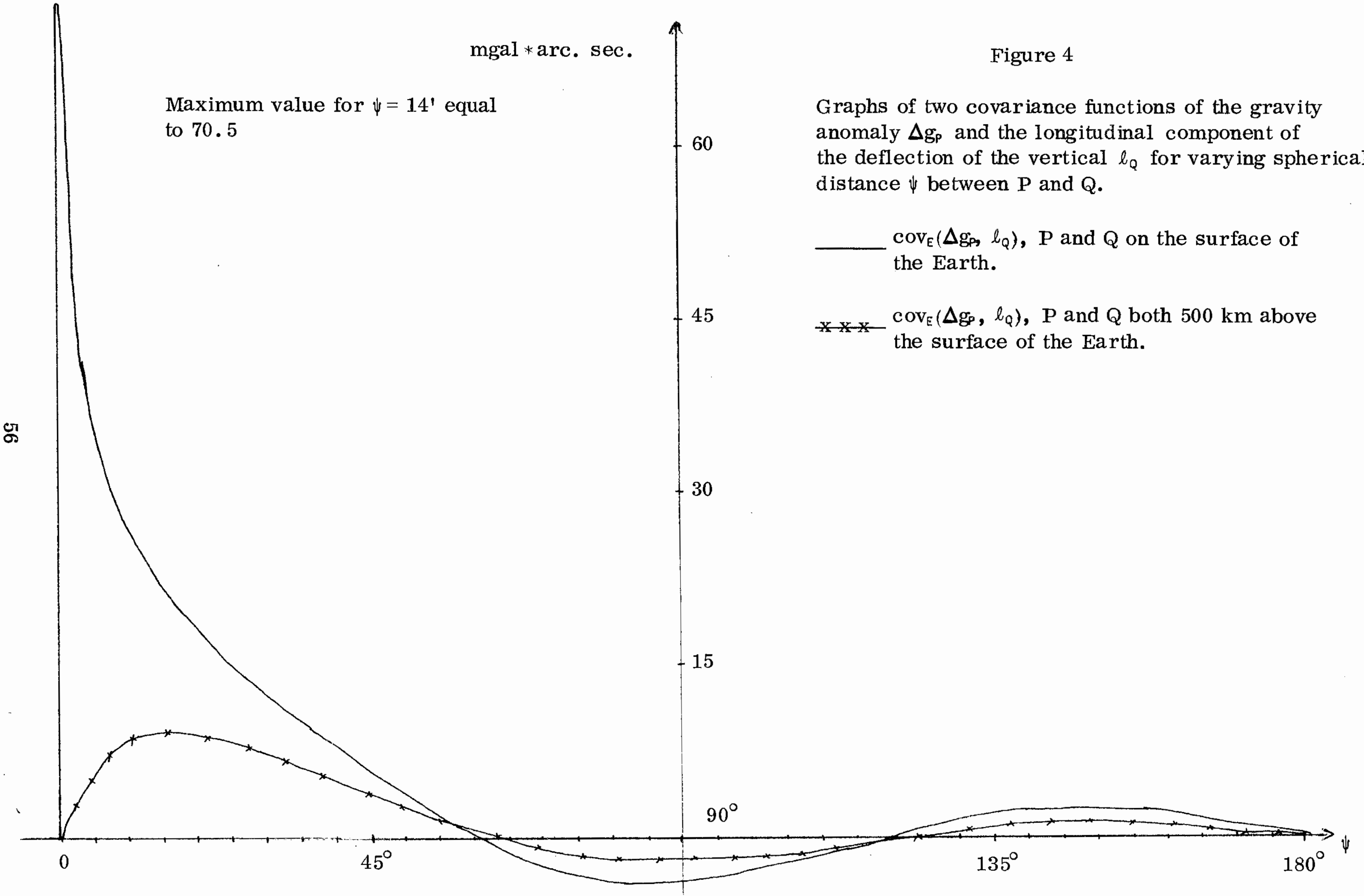



Figure 4

Graphs of two covariance functions of the gravity anomaly $\Delta g_P$ and the longitudinal component of the deflection of the vertical $\ell_Q$ for varying spherical distance $\psi$ between P and Q.

______ $\mathrm{cov}_E(\Delta g_P, \ell_Q)$, P and Q on the surface of the Earth.

-x-x-x- $\mathrm{cov}_E(\Delta g_P, \ell_Q)$, P and Q both 500 km above the surface of the Earth.

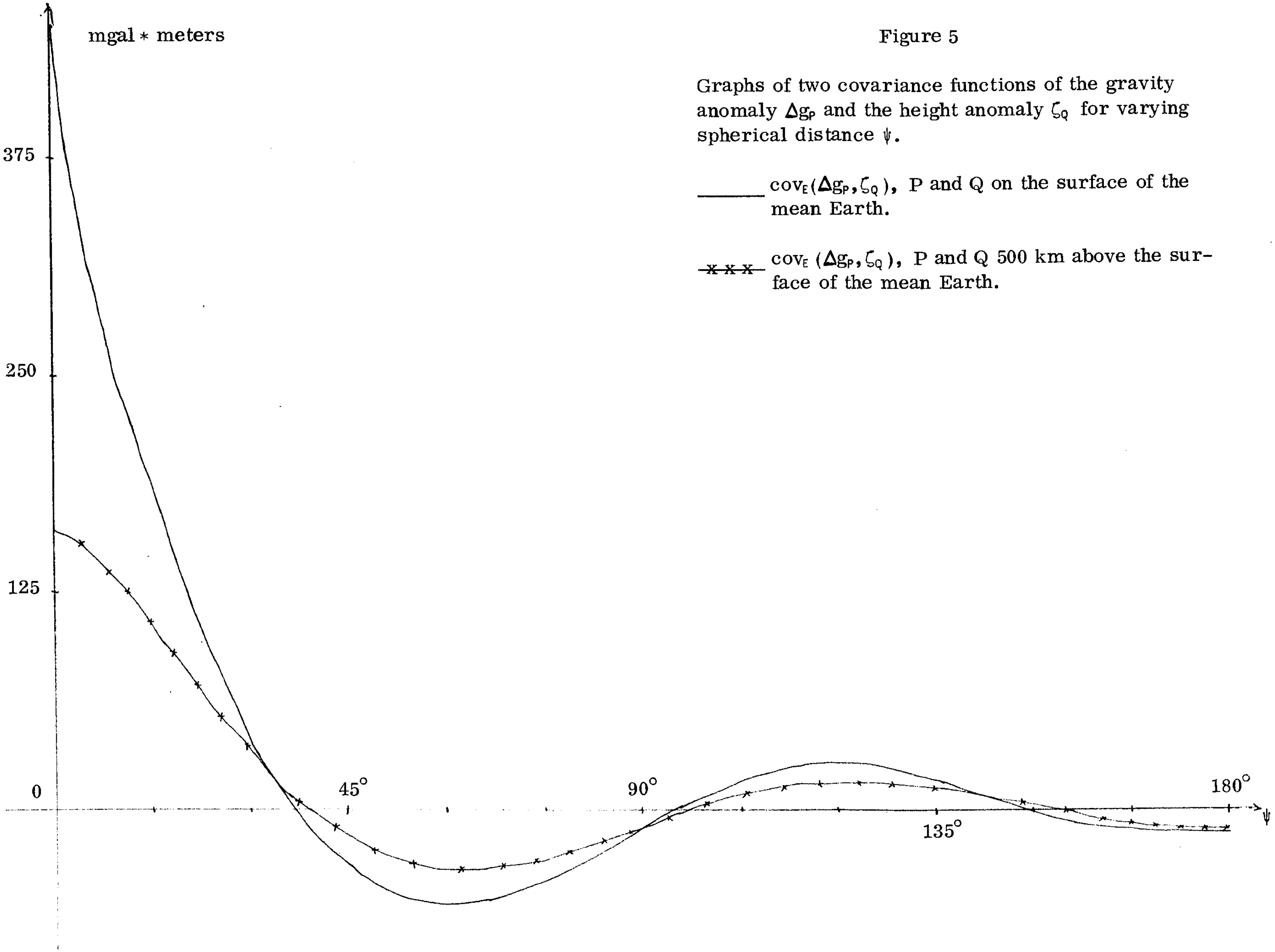



Figure 5

Graphs of two covariance functions of the gravity anomaly $\Delta g_P$ and the height anomaly $\zeta_Q$ for varying spherical distance $\psi$.

——— $cov_E(\Delta g_P, \zeta_Q)$, P and Q on the surface of the mean Earth.

-x-x-x- $cov_E(\Delta g_P, \zeta_Q)$, P and Q 500 km above the surface of the mean Earth.

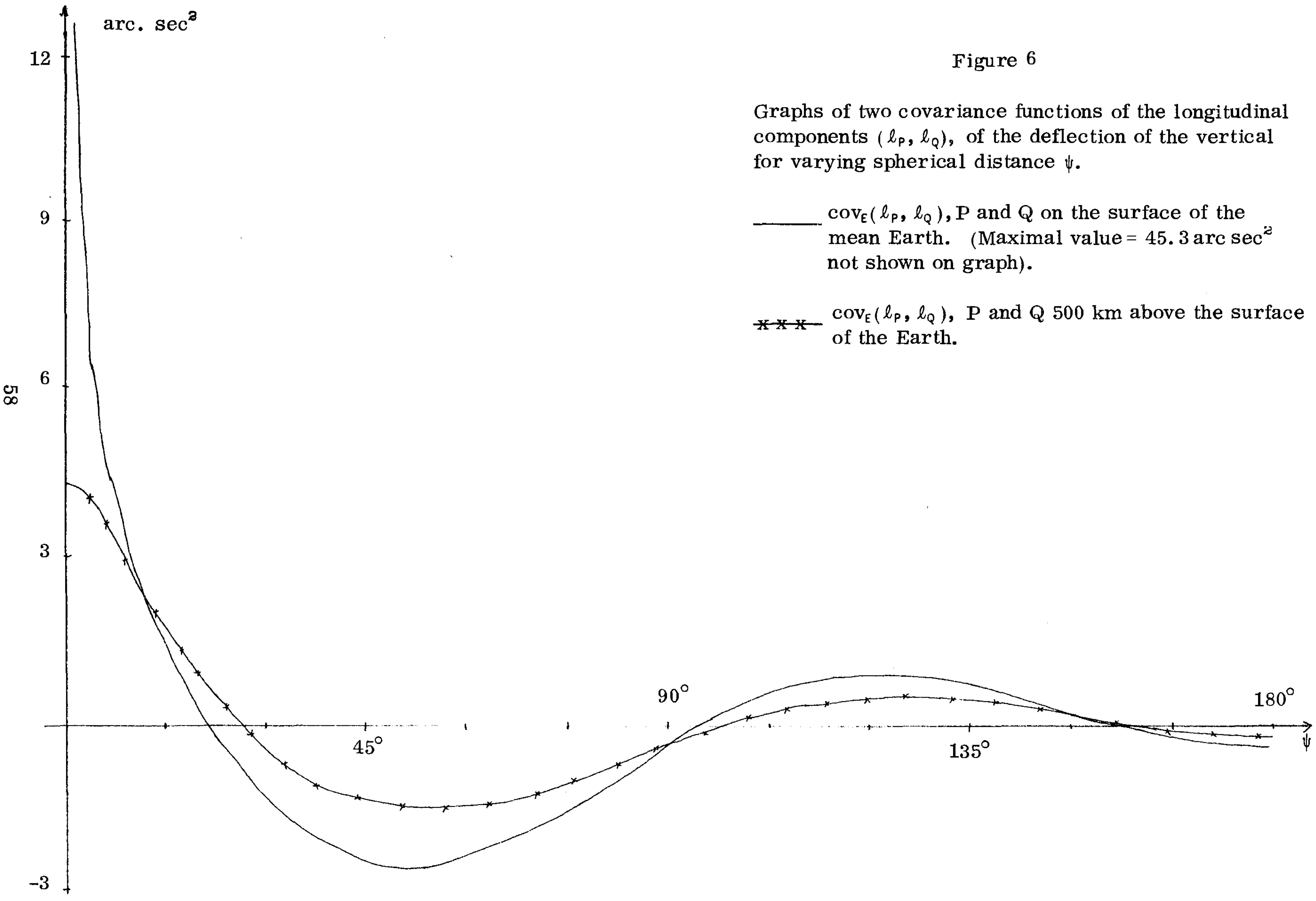



Figure 6

Graphs of two covariance functions of the longitudinal components ($\ell_P$, $\ell_Q$), of the deflection of the vertical for varying spherical distance $\psi$.

——— $\mathrm{cov}_E(\ell_P, \ell_Q)$, P and Q on the surface of the mean Earth. (Maximal value = 45.3 arc sec$^2$ not shown on graph).

-x-x-x- $\mathrm{cov}_E(\ell_P, \ell_Q)$, P and Q 500 km above the surface of the Earth.

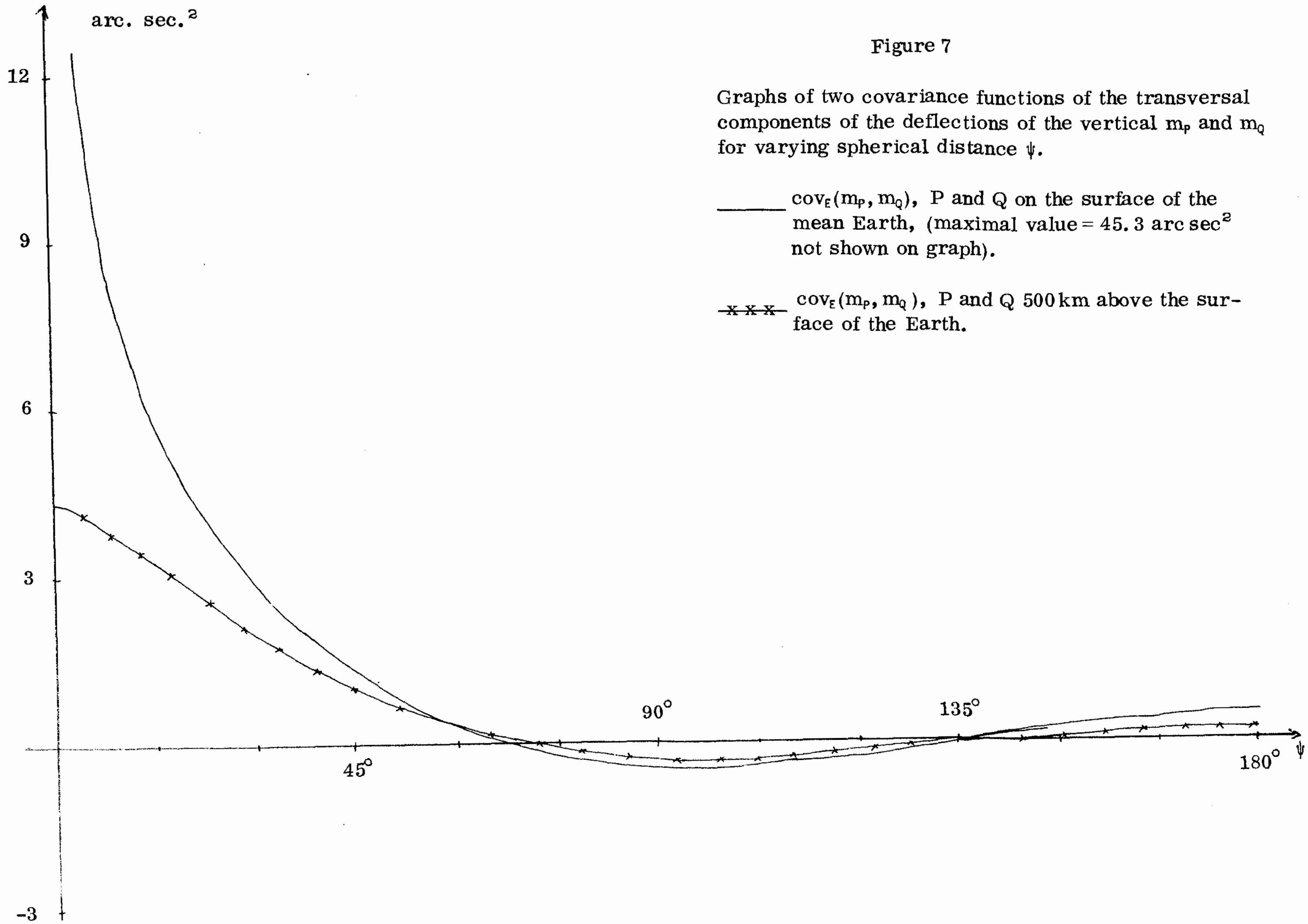



Figure 7

Graphs of two covariance functions of the transversal components of the deflections of the vertical $m_P$ and $m_Q$ for varying spherical distance $\psi$.

——— $cov_E(m_P, m_Q)$, P and Q on the surface of the mean Earth, (maximal value = 45.3 arc sec$^2$ not shown on graph).

-x-x-x- $cov_E(m_P, m_Q)$, P and Q 500 km above the surface of the Earth.

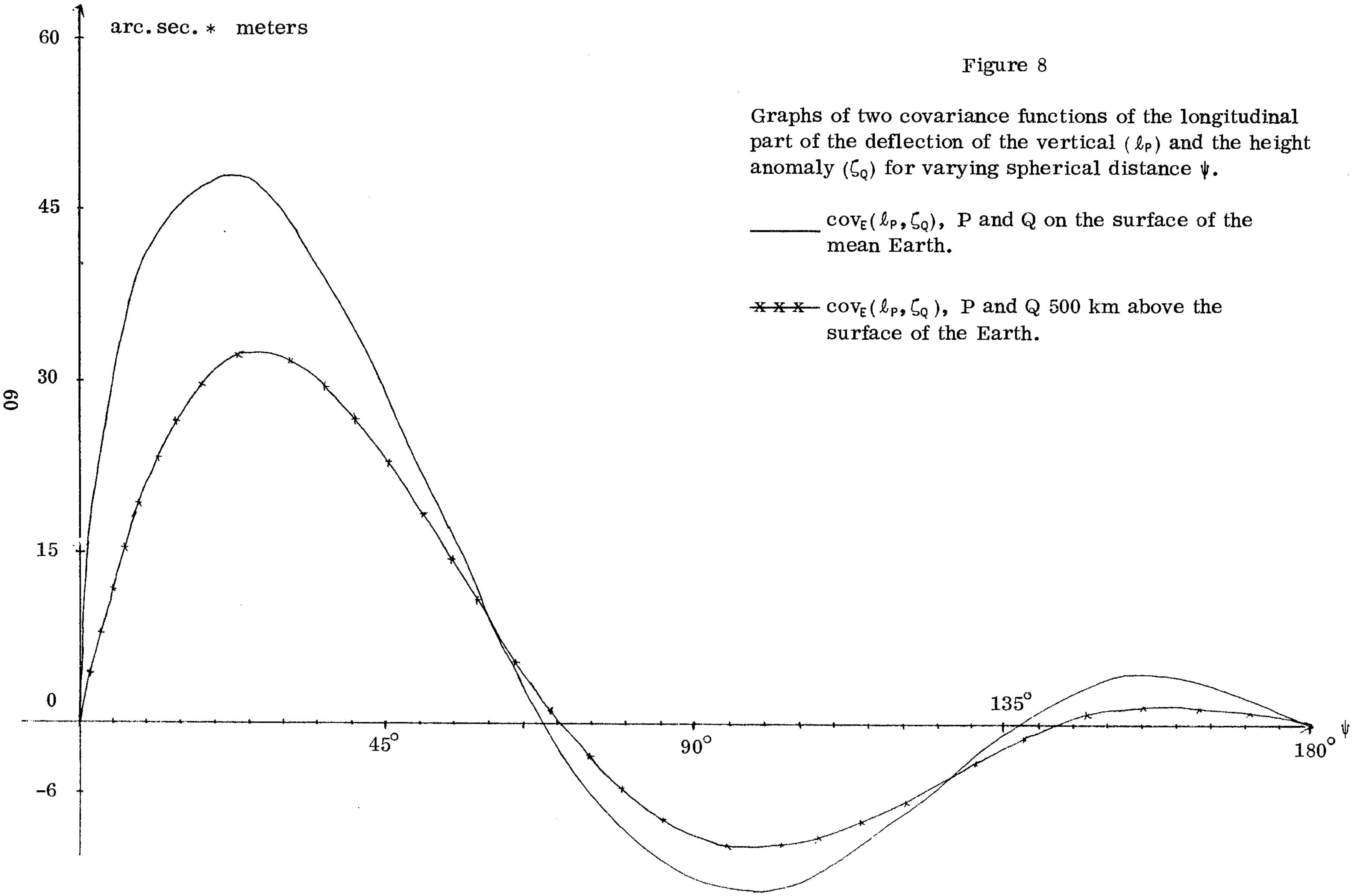



Figure 8

Graphs of two covariance functions of the longitudinal part of the deflection of the vertical ($\ell_P$) and the height anomaly ($\zeta_Q$) for varying spherical distance $\psi$.

——— $\mathrm{cov}_E(\ell_P, \zeta_Q)$, P and Q on the surface of the mean Earth.

-x-x-x- $\mathrm{cov}_E(\ell_P, \zeta_Q)$, P and Q 500 km above the surface of the Earth.

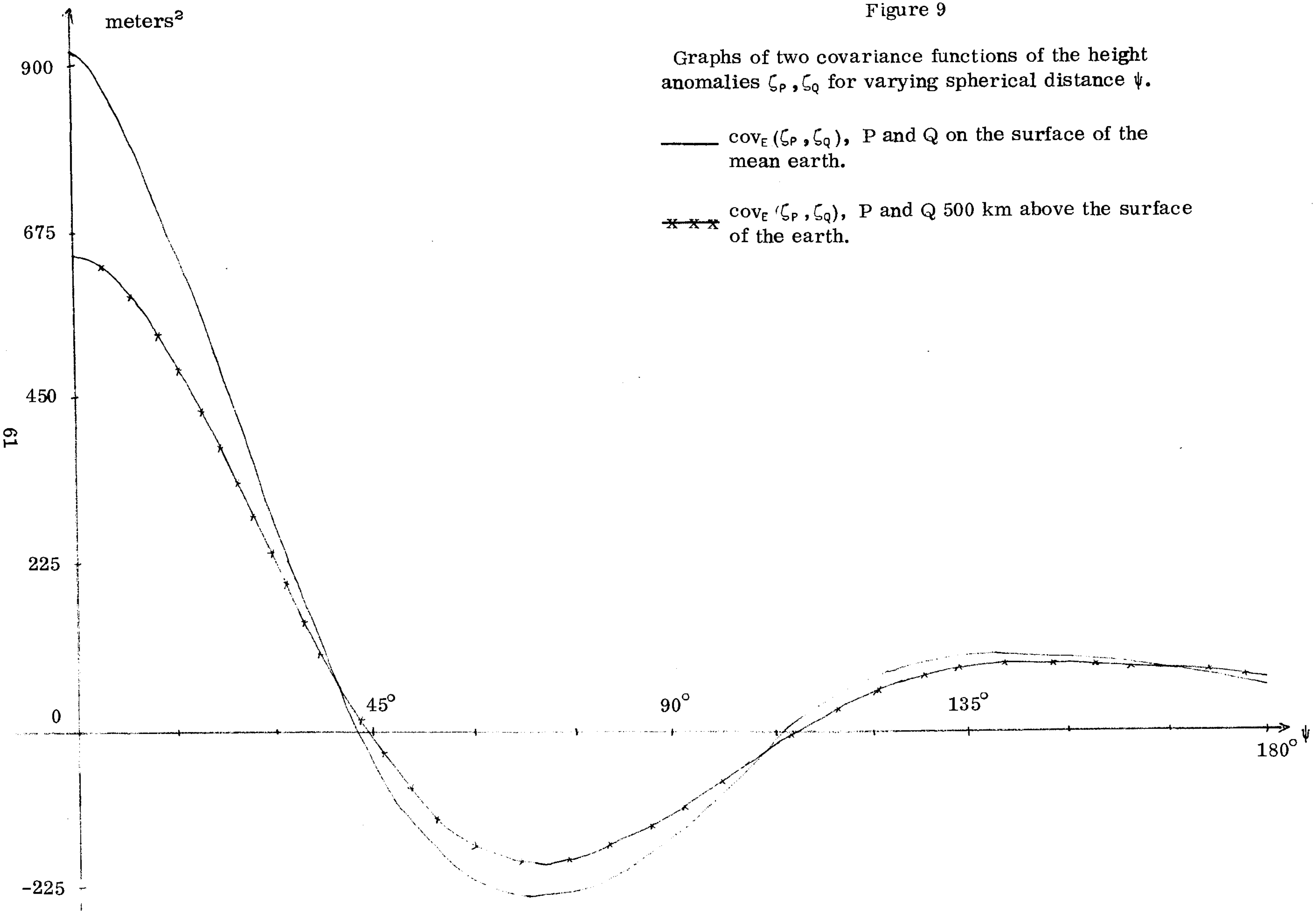



Figure 9

Graphs of two covariance functions of the height anomalies $\zeta_P, \zeta_Q$ for varying spherical distance $\psi$.

——— $\mathrm{cov}_E(\zeta_P, \zeta_Q)$, P and Q on the surface of the mean earth.

-x-x-x- $\mathrm{cov}_E(\zeta_P, \zeta_Q)$, P and Q 500 km above the surface of the earth.

## 9. Application of the Covariance Models for the Representation of Local Covariance Functions.

Local covariance functions of point or mean gravity anomalies may be estimated by formulas similar to (3) and (4) applied on the gravity data in a certain limited area. Thus, the anomalies must be centered, i.e. the mean value over the considered area will have to be subtracted.

Disregarding gravity information outside the considered area and subtraction of the local mean value correspond heuristically to disregarding the information contained in the low order harmonics.

We will here define a n'th order local (isotropic) covariance function as a covariance function, which can be derived from the covariance function of the anomalous potential (158) using the law of propagation of covariances:

$$\mathrm{cov}_k^n(T_P, T_Q) = K_k^n(P,Q) = \sum_{\ell=n+1}^{\infty} \sigma_{k,\ell}(T,T) s^{\ell+1} P_\ell(t) \tag{158}$$

where the superscript n is the order of the local covariance function and the subscript k is an integer used (as before) to distinguish between the different degree-variance models. Thus $K_k^n(P,Q)$ is in fact a special case of the models $\mathrm{cov}_\varepsilon(T_P, T_Q)$ considered above, having all degree-variances up to and inclusive of degree n equal to zero. We can then rewrite (158):

$$K_k^n(P,Q) = \sum_{\ell=0}^{\infty} \sigma_{k,\ell}(T,T) s^{\ell+1} P_\ell(t) - \sum_{\ell=0}^{n} \sigma_{k,\ell}(T,T) s^{\ell+1} P_\ell(t) \tag{158A}$$

$$= \sum_{\ell=0}^{n} (-\sigma_{k,\ell}(T,T)) s^{\ell+1} P_\ell(t) + K_k(P,Q)$$

For the quantity $\epsilon(T,T)$ defined in (151) we have:

$$\epsilon_\ell(T,T) = -\sigma_{k,\ell}(T,T)$$

and hence:

$$\epsilon_\ell(\Delta g, T) = -\sigma_{k,\ell}(\Delta g, T) \quad \text{and}$$

$$\epsilon_\ell(\Delta g, \Delta g) = -\sigma_{k,\ell}(\Delta g, \Delta g).$$

Then we can use the expressions (145)-(150) to write down the different covariance functions derived from (158):

$$\mathrm{cov}_k^n(\Delta g_P, T_Q) = \mathrm{cov}_k(\Delta g_P, T_Q) - \frac{R}{r}\sum_{\ell=0}^{n} \sigma_{k,\ell}(\Delta g, T) s^{\ell+1} P_\ell(t), \tag{159}$$

$$\mathrm{cov}_k^n(\Delta g_P, \Delta g_Q) = \mathrm{cov}_k(\Delta g_P, \Delta g_Q) - \sum_{\ell=0}^{n} \sigma_{k,\ell}(\Delta g, \Delta g) s^{\ell+2} P_\ell(t), \tag{160}$$

$$\begin{aligned}\mathrm{cov}_k^n(\ell_P, \ell_Q) = \mathrm{cov}_k(\ell_P, \ell_Q) - \Big(t \sum_{\ell=0}^{n} \sigma_{k,\ell}(T, T) s^{\ell+1} P'_\ell(t) \\ - \sin^2\psi \sum_{\ell=0}^{n} \sigma_{k,\ell}(T, T) s^{\ell+1} P''_\ell(t)\Big)/(G\cdot G'\cdot r\cdot r'),\end{aligned} \tag{161}$$

$$\mathrm{cov}_k^n(m_P, m_Q) = \mathrm{cov}_k(m_P, m_Q) - \Big(\sum_{\ell=0}^{n} \sigma_{k,\ell}(T, T) s^{\ell+1} P'_\ell(t)\Big)/(G\cdot G'\cdot r\cdot r') \tag{162}$$

$$\mathrm{cov}_k^n(\ell_P, \zeta_Q) = \mathrm{cov}_k(\ell_P, \zeta_Q) - \sin\psi\Big(\sum_{\ell=0}^{n} \sigma_{k,\ell}(T, T) s^{\ell+1} P'_\ell(t)\Big)/(G\cdot G'\cdot r) \tag{163}$$

and

$$\mathrm{cov}_k^n(\ell_P, \Delta g_Q) = \mathrm{cov}_k(\ell_P, \Delta g_Q) - \Big(\sum_{\ell=0}^{n} \sigma_{k,\ell}(\Delta g, T) s^{\ell+1} P'_\ell(t)\Big) \sin\psi\cdot R/(G\cdot r\cdot r') \tag{164}$$

The evaluation of the terms derived from the "global" covariance function $\mathrm{cov}_k(T_P, T_Q)$ have been explained in the preceding section. We will then have to evaluate the sums of the series (154), (155) and (156), which are series in the Legendre polynomials $P_\ell(t)$, their first derivatives $P'_\ell(t)$ and their second derivatives $P''_\ell(t)$, respectively.

This kind of series can be evaluated easily without explicitly evaluating the functions $P_\ell(t)$, $P'_\ell(t)$ and $P''_\ell(t)$. The technique is similar to the so called Horner-procedure for the evaluation of a usual polynomial:

$$\begin{aligned}\mathrm{Pol}(t) &= a_n t^n + a_{n-1} t^{n-1} + a_{n-2} t^{n-2} + \ldots + a_1 t + a_0 \\ &= (\ldots((a_n t + a_{n-1})t + a_{n-2})t + \ldots + a_1)t + a_0\end{aligned} \tag{165}$$

We can express this procedure through a recursion algorithm with terms:

$$b_\ell = b_{\ell+1} \cdot t + a_\ell \quad , \tag{166}$$

where the recursion starts with $b_{n+1}=0$ and where the value of Pol(t) is equal to the final recursion term $b_0$. The first, second (and higher order) derivatives of Pol(t) can be evaluated using recursion as well. The recursion formulas are found by differentiating (166)

$$b'_\ell = b'_{\ell+1} \cdot t + b_{\ell+1} \tag{167}$$

$$b''_\ell = b''_{\ell+1} \cdot t + 2b'_{\ell+1} \tag{168}$$

and the derivatives will be $Pol'(t) = b'_0$ and $Pol''(t) = b''_0$

This type of algorithm, which starts by accumulating the high order terms are especially useful when t is less than one, i.e. when a usual evaluation of $t^\ell$ and multiplication with $a_\ell$ contingently would add a small number to already accumulated terms. The essential point in the procedure is the simple fact that,

$$t^{\ell+1} - t \cdot t^\ell = 0$$

i.e., that there exists a recursion formula for the function $t^\ell$.

It is well known, that we have a simple recursion formula for the Legendre polynomials $P_\ell(t)$. By inspecting the formula for the covariance functions, we also note the term $s^{\ell+1}$ or $s^{\ell+2}$, which becomes smaller and smaller for $\ell$ increasing, because s is less than 1. So we can hope to find simple recursion formulas for the sums (154), (155) and (156), which furthermore should behave well numerically.

A general treatment of this type of summation problem is given in Clenshaw (1955, p. 118) (also valid for many other well known series as e.g. Chebyshev series or Neumann series of Bessel functions). He regards the sum of a series:

$$S_n = \sum_{\ell=0}^{n} a_\ell p_\ell(t) \tag{169}$$

for which there exists a three-term recursion formula between the functions $p_\ell(t)$:

$$p_{\ell+1}(t) + e_\ell p_\ell(t) + f_\ell p_{\ell-1}(t) = 0 \tag{170}$$

The coefficients $e_\ell$ and $f_\ell$ may be dependent of t as well as on $\ell$.

He proves, that the recursion algorithm:

$$b_\ell = -e_\ell b_{\ell+1} - f_{\ell+1} b_{\ell+2} + a_\ell \tag{171}$$

with $b_{n+2} = b_{n+1} = 0$, will furnish us with the sum (169), so that

$$S_n = b_0 p_0(t) + b_1 (p_1(t) + e_0 p_0(t)) \tag{172}$$

after n+1 recursion steps.

In the example above (165), we have $f_\ell = 0$, $e_\ell = -t$ and then we get from (171)

$$b_\ell = t \cdot b_{\ell+1} + a_\ell \quad \text{and} \quad S_n = b_0 + b_1 (t + (-t)) = b_0$$

as stated above in (166).

By differentiation of the recursion formula (171) and the formula (172) we get

$$b'_\ell = -e'_\ell b_{\ell+1} - e_\ell b'_{\ell+1} - f'_{\ell+1} b_{\ell+2} - f_{\ell+1} b'_{\ell+2} \, , \tag{173a}$$

$$b''_\ell = -e''_\ell b_{\ell+1} - e_\ell b''_{\ell+1} - 2e'_\ell b'_{\ell+1} - f''_{\ell+1} b_{\ell+2} - f_{\ell+1} b''_{\ell+2} - 2f'_{\ell+1} b'_{\ell+1} \tag{173b}$$

and

$$\begin{aligned} S'_n = {} & b'_0 p_0(t) + b_0 p'_0(t) + b'_1 (p_1(t) + e_0 p_0(t)) \\ & + b_1 (p'_1(t) + e'_0 p_0(t) + e_0 p'_0(t)) \text{ and} \end{aligned} \tag{173c}$$

$$\begin{aligned} S''_n = {} & b''_0 p_0(t) + b_0 p''_0(t) + 2b'_0 p'_0(t) + b''_1 (p_1(t) \cdot e_0 p_0(t)) \\ & + 2b'_0 (p'_1(t) + e'_0 p_0(t) + e_0 p'_0(t)) + b_1 (p''_1(t) + e''_0 p_0(t) + e_0 p''_0(t) + 2e'_0 p'_0(t)). \end{aligned} \tag{173d}$$

For the Legendre polynomials we have the well known recursion formula:

$$P_{\ell+1}(t) - \frac{2\ell+1}{\ell+1} \cdot t \cdot P_\ell(t) + \frac{\ell}{\ell+1} P_{\ell-1}(t) = 0 \tag{174}$$

Thus, by multiplying (174) with $s^{\ell+2}$ we get:

$$s^{\ell+2}P_{\ell+1}(t) - \frac{2\ell+1}{\ell+1}\, t\cdot s(s^{\ell+1}\cdot P_\ell(t)) + \frac{\ell\cdot s^2}{\ell+1}(s^\ell P_\ell(t)) = 0\,, \tag{175}$$

and thereby in fact a recursion formula for the functions

$$p_\ell(t) = s^{\ell+1}P_\ell(t),$$

which then directly can be applied on the series (154)-(156).

The quantities $e_\ell$ and $f_\ell$ in (170) becomes:

$$e_\ell = -\frac{2\ell+1}{\ell+1}\cdot t\cdot s \quad \text{and} \tag{176a}$$

$$f_\ell = \frac{\ell\cdot s^2}{\ell+1} \tag{176b}$$

Using (176) and that $p_o(t) = s\cdot P_o(t) = s$ and $p_1(t) = s^2 t$, we get:

$$e'_\ell = -\frac{2\ell+1}{\ell+1}\cdot s, \tag{177}$$

$$e''_\ell = 0 \tag{178}$$

$$f'_\ell = f''_\ell = 0 \tag{179}$$

$$p'_o(t) = p''_o(t) = 0 \tag{180}$$

$$p'_1(t) = s^2 \quad \text{and} \tag{181}$$

$$p''_1(t) = 0 \tag{182}$$

Then by (177)-(182) and (171)-(173) we get the following recursion formula for the quantities (154)-(156)(with $a_\ell$ equal to $\sigma_{k,\ell}(T,T)$, $\sigma_{k,\ell}(\Delta g, T)$ or $\sigma_{k,\ell}(\Delta g, \Delta g)\cdot s$ respectively):

$$b_\ell = -e_\ell b_{\ell+1} - f_{\ell+1} b_{\ell+2} + a_\ell \tag{183}$$

$$= \frac{2\ell+1}{\ell+1} \cdot t \cdot s \cdot b_{\ell+1} - \frac{(\ell+1)\cdot s^2}{(\ell+2)} b_{\ell+2} + a_\ell \ ,$$

$$S_n = b_0 \cdot s + b_1(s^2 t - (st)s) = b_0 \cdot s \ , \tag{184}$$

$$b'_\ell = \frac{2\ell+1}{\ell+1} \cdot s \cdot b_{\ell+1} + \frac{2\ell+1}{\ell+1} \cdot t \cdot s \cdot b'_{\ell+1} - \frac{(\ell+1)}{\ell+2} s^2 \cdot b'_{\ell+2}$$

$$= \frac{2\ell+1}{\ell+1} s \left(b_{\ell+1} + t \cdot b'_{\ell+1}\right) - \frac{(\ell+1)s^2}{(\ell+2)} b'_{\ell+2} \tag{185}$$

$$S'_n = b'_0 \cdot s \tag{186}$$

and finally:

$$b''_\ell = \frac{2\ell+1}{\ell+1} s \cdot \left(2b'_{\ell+1} + t \cdot b''_{\ell+1}\right) - \frac{(\ell+1)\, s^2}{(\ell+2)} b''_{\ell+2}, \text{ with} \tag{187}$$

$$S''_n = b''_0 \cdot s \tag{188}$$

We would like to point out, that the recursion formulas (183)-(188) are valid for the computation of sums of a usual Legendre-series. The formulas can be obtained from equations (183)-(188) simply by putting s equal to one.

The subroutine presented in the appendix has been used to compute $cov_4^{20}(\Delta g_P, \ell_Q)$, $cov_4^{20}(\Delta g_P, \zeta_Q)$, $cov_4^{20}(\ell_P, \ell_Q)$, $cov_4^{20}(m_P, m_Q)$, $cov_4^{20}(\ell_P, \zeta_Q)$ and $cov_4^{20}(\zeta_P, \zeta_Q)$ for spherical distance $\psi$ varying with $\frac{1}{2}^\circ$ increments from $0^\circ$ to $25^\circ$. The values are shown in table 11. (The degree-variance model defined by the constants given in Table 7 has again been used.)

The analytic local covariance functions model discussed above can be used to find approximations for the empirical determined local covariance functions. Such a

Table 11
Covariances between various quantities computed from the local 20th order covariance functions using the anomaly degree variances of model 4.

Covariances Between

| $\psi$ | $\Delta g_P, \Delta g_Q$ mgal$^2$ | $\Delta g_P, \ell_Q$ mgal·arc sec | $\Delta g_P, \zeta_Q$ mgal·m | $\ell_P, \ell_Q$ arc sec$^2$ | $m_P, m_Q$ arc sec$^2$ | $\ell_P, \zeta_Q$ arc sec·m | $\zeta_P, \zeta_Q$ m$^2$ |
|---|---|---|---|---|---|---|---|
| 0° 0.0' | 1519.6 | 0.0 | 88.3 | 34.5 | 34.5 | 0.0 | 13.3 |
| 0 30.0 | 527.1 | 64.2 | 71.2 | 8.5 | 16.3 | 4.4 | 12.6 |
| 1 0.0 | 300.1 | 53.8 | 55.3 | 3.5 | 11.0 | 5.9 | 11.2 |
| 1 30.0 | 183.5 | 45.2 | 42.0 | 1.0 | 8.0 | 6.5 | 9.5 |
| 2 0.0 | 111.2 | 37.9 | 30.9 | -0.5 | 6.1 | 6.5 | 7.7 |
| 2 30.0 | 62.6 | 31.5 | 21.5 | -1.4 | 4.6 | 6.3 | 6.0 |
| 3 0.0 | 28.6 | 26.0 | 13.8 | -2.0 | 3.6 | 5.8 | 4.4 |
| 3 30.0 | 4.7 | 21.1 | 7.5 | -2.4 | 2.8 | 5.2 | 2.9 |
| 4 0.0 | -12.2 | 16.8 | 2.3 | -2.6 | 2.1 | 4.5 | 1.6 |
| 4 30.0 | -23.7 | 13.0 | -1.7 | -2.7 | 1.6 | 3.8 | 0.4 |
| 5 0.0 | -31.2 | 9.6 | -4.7 | -2.6 | 1.2 | 3.1 | -0.5 |
| 5 30.0 | -35.4 | 6.7 | -6.9 | -2.5 | 0.8 | 2.4 | -1.2 |
| 6 0.0 | -37.2 | 4.1 | -8.3 | -2.4 | 0.5 | 1.7 | -1.8 |
| 6 30.0 | -37.0 | 1.9 | -9.1 | -2.2 | 0.3 | 1.1 | -2.2 |
| 7 0.0 | -35.4 | 0.1 | -9.4 | -1.9 | 0.2 | 0.6 | -2.4 |
| 7 30.0 | -32.6 | -1.4 | -9.2 | -1.7 | 0.0 | 0.1 | -2.5 |
| 8 0.0 | -29.0 | -2.6 | -8.7 | -1.4 | -0.1 | -0.3 | -2.5 |
| 8 30.0 | -24.8 | -3.5 | -7.9 | -1.1 | -0.1 | -0.6 | -2.3 |
| 9 0.0 | -20.4 | -4.1 | -6.8 | -0.9 | -0.2 | -0.9 | -2.1 |
| 9 30.0 | -15.8 | -4.5 | -5.7 | -0.6 | -0.2 | -1.1 | -1.9 |
| 10 0.0 | -11.2 | -4.7 | -4.4 | -0.4 | -0.2 | -1.2 | -1.5 |
| 10 30.0 | -6.9 | -4.7 | -3.2 | -0.1 | -0.2 | -1.3 | -1.2 |
| 11 0.0 | -2.8 | -4.5 | -1.9 | 0.1 | -0.2 | -1.3 | -0.8 |
| 11 30.0 | 0.8 | -4.2 | -0.8 | 0.2 | -0.2 | -1.3 | -0.5 |
| 12 0.0 | 4.1 | -3.8 | 0.3 | 0.4 | -0.2 | -1.2 | -0.2 |
| 12 30.0 | 6.8 | -3.3 | 1.3 | 0.5 | -0.2 | -1.1 | 0.2 |
| 13 0.0 | 8.9 | -2.7 | 2.1 | 0.6 | -0.1 | -0.9 | 0.4 |
| 13 30.0 | 10.6 | -2.1 | 2.7 | 0.6 | -0.1 | -0.8 | 0.7 |
| 14 0.0 | 11.6 | -1.5 | 3.2 | 0.7 | -0.1 | -0.6 | 0.8 |
| 14 30.0 | 12.2 | -0.9 | 3.5 | 0.7 | -0.1 | -0.4 | 1.0 |
| 15 0.0 | 12.3 | -0.4 | 3.7 | 0.7 | -0.0 | -0.2 | 1.1 |
| 15 30.0 | 12.0 | 0.2 | 3.7 | 0.6 | -0.0 | -0.1 | 1.1 |
| 16 0.0 | 11.3 | 0.7 | 3.6 | 0.6 | 0.0 | 0.1 | 1.1 |
| 16 30.0 | 10.2 | 1.1 | 3.4 | 0.5 | 0.0 | 0.2 | 1.1 |
| 17 0.0 | 8.9 | 1.4 | 3.0 | 0.4 | 0.0 | 0.4 | 1.0 |
| 17 30.0 | 7.4 | 1.7 | 2.6 | 0.3 | 0.0 | 0.5 | 0.9 |
| 18 0.0 | 5.8 | 1.9 | 2.1 | 0.2 | 0.1 | 0.5 | 0.8 |
| 18 30.0 | 4.0 | 2.0 | 1.6 | 0.2 | 0.1 | 0.6 | 0.6 |
| 19 0.0 | 2.3 | 2.0 | 1.1 | 0.1 | 0.1 | 0.6 | 0.4 |
| 19 30.0 | 0.6 | 2.0 | 0.6 | -0.0 | 0.1 | 0.6 | 0.3 |
| 20 0.0 | -0.9 | 1.9 | 0.0 | -0.1 | 0.1 | 0.6 | 0.1 |
| 20 30.0 | -2.4 | 1.7 | -0.5 | -0.2 | 0.1 | 0.6 | -0.1 |
| 21 0.0 | -3.6 | 1.5 | -0.9 | -0.2 | 0.0 | 0.5 | -0.2 |
| 21 30.0 | -4.7 | 1.3 | -1.3 | -0.3 | 0.0 | 0.5 | -0.3 |
| 22 0.0 | -5.5 | 1.0 | -1.6 | -0.3 | 0.0 | 0.4 | -0.4 |
| 22 30.0 | -6.1 | 0.7 | -1.8 | -0.3 | 0.0 | 0.3 | -0.5 |
| 23 0.0 | -6.5 | 0.5 | -2.0 | -0.3 | 0.0 | 0.2 | -0.6 |
| 23 30.0 | -6.6 | 0.2 | -2.1 | -0.3 | 0.0 | 0.1 | -0.6 |
| 24 0.0 | -6.4 | -0.1 | -2.1 | -0.3 | 0.0 | 0.0 | -0.7 |
| 24 30.0 | -6.1 | -0.4 | -2.0 | -0.3 | -0.0 | -0.1 | -0.6 |
| 25 0.0 | -5.6 | -0.6 | -1.9 | -0.3 | -0.0 | -0.1 | -0.6 |

covariance function (of e.g. point gravity anomalies) differ from a global covariance function by having another (generally smaller) value for spherical distance $\psi$ equal to zero and by having its first zero point occurring for a much smaller spherical distance. We will denote this distance by $\psi_1$, i.e. $cov_n^n(\Delta g_P, \Delta g_Q) = 0$ for the spherical distance between P and Q equal to $\psi_1$ and all points P and Q with smaller spherical distance will have a positive covariance.

Note in Table 11, that the $\psi_1$ value is equal to $3^\circ 37'$. The first zero point for $cov_E(\Delta g_P, \Delta g_Q)$ was (cf. Table 9 ) equal to $29^\circ$. It is a general trend (which can be verified for the here discussed degree-covariance models by computational experiments), that the first zero point $\psi_1$ occurs at decreasing $\psi$ values for increasing order of the local covariance function. Table 12 shows the value of $\psi_1$ for $cov_4^n(\Delta g_P, \Delta g_Q)$ for various n values. Note in the table, that the first zero point will occur between $\psi = 0$ and $\psi = 90^\circ/n$.

Table 12

The spherical distance of the first zero point ($\psi_1$) for some n'th order local covariance functions of gravity anomalies. The degree-variance model used is given by the constants of Table 7.

| Order (n) | $\psi_1$ | Order (n) | $\psi_1$ |
|---|---|---|---|
| 20 | $3^\circ 37'$ | 140 | 35' |
| 40 | $2^\circ 55'$ | 160 | 30' |
| 60 | $1^\circ 18'$ | 180 | 27' |
| 80 | 59' | 200 | 25' |
| 100 | 48' | 220 | 22' |
| 120 | 40' | 240 | 21' |

By inspecting the graph of an empirically estimates local covariance function it is generally possible to find it's first zero point. The corresponding order of the local covariance function can hence be estimated by determining a n greater that $90^\circ/\psi_1$ for which the two zero points are as near to each other as possible. The local covariance function $cov_k^n(\Delta g_P, \Delta g_Q)$ can then be fitted to the estimated covariance function by multiplying the degree-variances of the adopted model by the ratio between the empirical determined variance and the value of $cov_k^n(\Delta g_P, \Delta g_Q)$ (i.e. the value for $\psi = 0^\circ$).

## 10. Representation of covariance functions of mean gravity anomalies.

We will in this section regard the covariance function of mean gravity anomalies and discuss a representation of these by a certain related point gravity covariance function.

In section 2 above we described how covariance functions of different kinds of mean gravity anomalies can be represented by a covariance function of mean gravity anomalies, meaned over a spherical cap.

The relation between the degree-variances of this spherical cap mean gravity covariance function and the degree-variances of the point anomaly covariance is (cf. equation (11)):

$$c_\ell(\overline{\Delta g}, \overline{\Delta g}) = \beta_\ell^2\, c_\ell = \beta_\ell^2 \cdot \sigma_\ell(\Delta g, \Delta g) \tag{189}$$

where the quantities $\beta_\ell$ are given by equation (12). From this equation we have that

$$\beta_\ell = \frac{1}{1-\cos\psi_0} \cdot \frac{1}{2\ell+1} \left[ P_{\ell-1}(\cos\psi_0) - P_{\ell+1}(\cos\psi_0) \right]$$

$$\leq \frac{1}{1-\cos\psi_0} \cdot \frac{1}{2\ell+1} \cdot 2$$

because $P_\ell(\cos\psi_0)$ is less than or equal to one for all $\psi_0$.

Hence (for $\psi_0 \neq 0$):

$$\lim_{\ell\to\infty} \beta_\ell = 0.$$

Therefore it is not necessary to carry out the summation of the series representing $\overline{C}(P, Q)$ to the same height degree as for the series representing $C(P, Q)$. The recursion formula (172) may in this case, be well suited for computation of mean anomaly covariance values.

Unfortunately, none of the degree-variance models (65)-(69) result in closed expressions for $\overline{C}(P, Q)$. But we may get an intuitive feeling of how a possible representation can be obtained by regarding the graphs of the two point anomaly covariance functions in Figure 3 and compare these with the graph of the mean anomaly covariance function in Figure one. The graphs of the mean anomaly covariance function will either lie in between or near the graphs of the two point anomaly covariance functions. In fact, by varying the height of the points P and Q, points $Q_1$ and $Q_2$ can be found for which the anomaly covariance function $C(Q_1, Q_2)$ gives a good approximation to e.g. the $1^\circ \times 1^\circ$ mean anomaly covariance function. Table 13 gives the mean square variation of the point anomalies for some values of the height of $Q_1$ ($h_{Q_1}$) and $Q_2$ ($h_{Q_2}$) above the surface of the Earth. (The values have been computed using the subroutine presented in the appendix).

Table 13

Table of the point anomaly variance $C(Q_1,Q_1)$ for different heights $h_{Q_1}$ equal to $h_{Q_2}$

| $h_{Q_1}$ km | $C(Q_1, Q_1)$ mgal² | $h_{Q_1}$ km | $C(Q_1, Q_1)$ mgal² |
|---|---|---|---|
| 0 | 1795 | 80.0 | 343 |
| 2.5 | 1346 | 160.0 | 207 |
| 5.0 | 1148 | 320.0 | 108 |
| 10.0 | 931 | 640.0 | 46 |
| 20.0 | 715 | 1280.0 | 14 |
| 40.0 | 515 | | |

The height, $h_Q$, corresponding to the value $\bar{C}(P, P) = 919.66$ for the $1^\circ \times 1^\circ$ mean anomaly covariance (Table One) has been estimated to be 10.4 km.

For the point anomaly covariance functions for points $Q_1$ and $Q_2$ in this height we have:

$$C(Q_1, Q_2) = \sum_{\ell=2}^{\infty} c_\ell \left(\frac{R}{R_e + h_Q}\right)^{2\ell+4} P_\ell(t) = \sum_{\ell=2}^{\infty} c_\ell \left(\frac{R}{R_e}\right)^{2\ell+4} \left(\frac{R_e}{R_e + h_Q}\right)^{2\ell+4} P_\ell(t) \tag{190}$$

In using $C(Q_1, Q_2)$ as a representation for $\bar{C}(P, Q)$, where P and Q are on the surface of the Earth, we are approximating

$$C(\overline{\Delta g}_P, \overline{\Delta g}_Q) = \sum_{\ell=2}^{\infty} c_\ell \beta_\ell^2 \left(\frac{R}{R_e}\right)^{2\ell+4} P_\ell(t) \tag{191}$$

by

$$\sum_{\ell=2}^{\infty} c_\ell \left(\frac{R_e}{R_e + h_Q}\right)^{2\ell+4} \left(\frac{R}{R_e}\right)^{2\ell+4} P_\ell(t),$$

i.e. we are approximating

$$\beta_\ell^2 \text{ by } \left(\frac{R_e}{R_e + h_Q}\right)^{2\ell+4}$$

In Table 14 values of $\beta_\ell^2$ and $\left(\frac{R_e}{R_e + h_Q}\right)^{2\ell+4}$ are presented corresponding to the $1^\circ \times 1^\circ$ mean anomaly covariance function. The values of $\beta_\ell^2$ has been obtained by squaring the values given in Table B of the appendix.

Table 14

Values of $\beta_\ell^2$ and $\left(\frac{R_e}{R_e + h_Q}\right)^{2\ell+4}$ for $h_Q = 10.4$ km and $\psi_0 = 0^\circ.564$

| $\ell$ | $\beta_\ell^2$ | $\left(\frac{R_e}{R_e + h_Q}\right)^{2\ell+4}$ | $\ell$ | $\beta_\ell^2$ | $\left(\frac{R_e}{R_e + h_Q}\right)^{2\ell+4}$ |
|---|---|---|---|---|---|
| 2 | 0.999 | 0.987 | 60 | 0.915 | 0.817 |
| 10 | 0.997 | 0.961 | 70 | 0.885 | 0.791 |
| 20 | 0.990 | 0.931 | 80 | 0.853 | 0.765 |
| 30 | 0.978 | 0.901 | 90 | 0.817 | 0.741 |
| 40 | 0.961 | 0.872 | 100 | 0.779 | 0.717 |
| 50 | 0.940 | 0.844 | 110 | 0.738 | 0.694 |

Table 14 shows the similarity between the $\beta_\ell^2$ terms and the $(R_e/(R_e+h_Q))$ terms for the specific $\psi_0$ and $h_Q$ chosen.

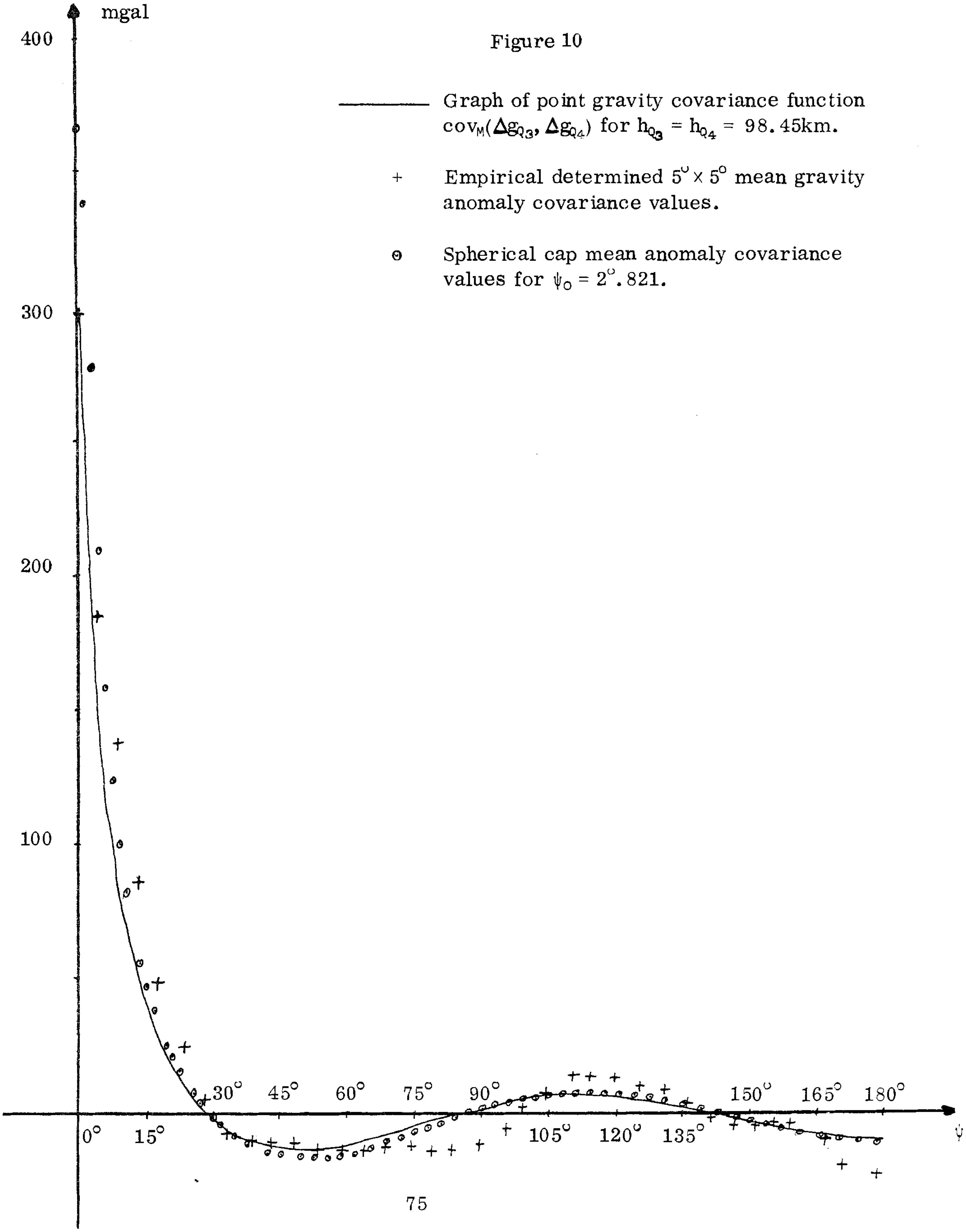
Table 15 gives values of (1) the empirical $1^\circ$ equal area mean gravity anomaly covariance function as taken from Table one, and designated as cov $(\overline{\Delta g_P}, \overline{\Delta g_Q})$, (2) the point gravity and point height anomaly covariance functions $\mathrm{cov}_M(\Delta g_{Q_1}, \Delta g_{Q_2})$, $\mathrm{cov}_M(\zeta_{Q_1}, \zeta_{Q_2})$ for $h_{Q_1} = h_{Q_2} = 10.4$ km and (3) the (circular cap, $\psi_0 = 0^\circ.564$) mean gravity and height anomaly covariance functions $\mathrm{cov}_M(\overline{\Delta g_P}, \overline{\Delta g_Q})$, $\mathrm{cov}_M(\overline{\zeta_P}, \overline{\zeta_Q})$. The subscript M indicates, that we have used the anomaly degree variance model of table seven, with $\sigma_2(\Delta g, \Delta g) = 7.5$ mgal$^2$. The table shows a reasonable good argument between the empirical determined covariance function and the two functions $\mathrm{cov}_M(\overline{\Delta g_P}, \overline{\Delta g_Q})$ and $\mathrm{cov}_M(\Delta g_{Q_1}, \Delta g_{Q_2})$. We also see, that it is reasonable to use the point height anomaly covariance function $\mathrm{cov}_M(\zeta_{Q_1}, \zeta_{Q_2})$ for the representation of the mean height anomaly.

Table 15

Values of the empirical $1^\circ \times 1^\circ$ mean gravity anomaly covariance function and related point and (spherical cap) mean gravity and height anomaly covariance functions.

| $\psi$ | $cov(\overline{\Delta g}_P, \overline{\Delta g}_Q)$ $mgal^2$ | $cov_M(\Delta g_{Q_1}, \Delta g_{Q_2})$ $mgal^2$ | $cov_M(\overline{\Delta g}_P, \overline{\Delta g}_Q)$ $mgal^2$ | $cov_M(\zeta_{Q_1}, \zeta_{Q_2})$ $m^2$ | $cov_M(\overline{\zeta}_P, \overline{\zeta}_Q)$ $m^2$ |
|---|---|---|---|---|---|
| 0.0° | 919.7 | 919.7 | 848.0 | 916.8 | 926.2 |
| 0.5 | 671.6 | 698.2 | 749.5 | 915.9 | 925.3 |
| 1.0 | 493.4 | 530.9 | 577.8 | 913.7 | 923.0 |
| 1.5 | 368.2 | 429.2 | 455.7 | 910.3 | 919.5 |
| 2.0 | 285.4 | 360.6 | 377.4 | 906.2 | 915.1 |
| 2.5 | 236.1 | 310.4 | 322.2 | 901.2 | 909.9 |
| 3.0 | 211.4 | 272.0 | 280.7 | 895.7 | 904.2 |
| 3.5 | 200.7 | 241.5 | 248.3 | 889.5 | 897.8 |
| 4.0 | 193.4 | 216.7 | 222.1 | 882.9 | 891.0 |
| 5.0 | 155.9 | 178.5 | 182.2 | 868.2 | 875.9 |
| 6.0 | 141.4 | 150.3 | 153.0 | 851.8 | 859.2 |
| 8.0 | 117.4 | 111.3 | 112.9 | 815.0 | 821.7 |
| 10.0 | 96.5 | 85.1 | 86.2 | 773.9 | 779.8 |
| 12.0 | 74.6 | 66.2 | 66.9 | 729.2 | 734.6 |
| 14.0 | 59.8 | 51.7 | 52.2 | 681.9 | 686.7 |
| 16.0 | 46.0 | 40.2 | 40.6 | 632.5 | 636.8 |
| 18.0 | 37.0 | 30.9 | 31.1 | 581.7 | 585.5 |
| 20.0 | 29.3 | 23.1 | 23.2 | 530.1 | 533.4 |
| 22.0 | 21.6 | 16.5 | 16.5 | 478.0 | 480.8 |
| 24.0 | 11.8 | 10.9 | 10.9 | 425.9 | 428.3 |
| 26.0 | 6.6 | 6.1 | 6.0 | 374.2 | 376.2 |
| 28.0 | 0.5 | 2.0 | 1.9 | 323.3 | 324.9 |
| 30.0 | -3.3 | -1.6 | 1.7 | 273.6 | 274.8 |
| 35.0 | -12.7 | -8.3 | -8.6 | 156.0 | 156.4 |
| 40.0 | -15.4 | -12.7 | -12.9 | 51.2 | 50.9 |
| 45.0 | -11.9 | -15.1 | -15.4 | -38.0 | -38.8 |
| 50.0 | -17.9 | -16.0 | -16.3 | -109.9 | -111.0 |
| 55.0 | -17.4 | -15.7 | -15.9 | -163.6 | -164.9 |
| 60.0 | -12.5 | -14.3 | -14.5 | -199.4 | -200.8 |
| 65.0 | -9.1 | -12.3 | -12.4 | -218.0 | -219.4 |
| 70.0 | -8.8 | -9.7 | -9.8 | -221.2 | -222.5 |
| 75.0 | -6.1 | -6.9 | -6.9 | -210.9 | -212.0 |
| 80.0 | -5.9 | -3.9 | -3.9 | -189.6 | -190.5 |
| 85.0 | -6.0 | -1.1 | -1.1 | -160.0 | -160.6 |
| 90.0 | -1.8 | 1.6 | 1.6 | -124.7 | -125.1 |
| 95.0 | 1.5 | 3.8 | 3.9 | -86.5 | -86.6 |
| 100.0 | 8.0 | 5.6 | 5.7 | -47.7 | -47.6 |
| 105.0 | 9.4 | 6.8 | 6.9 | -10.5 | -10.2 |
| 110.0 | 9.2 | 7.5 | 7.6 | 23.2 | 23.7 |
| 115.0 | 10.5 | 7.6 | 7.7 | 54.8 | 52.8 |
| 120.0 | 7.0 | 7.1 | 7.2 | 75.4 | 76.1 |
| 125.0 | 5.6 | 4.6 | 6.2 | 93.9 | 93.2 |
| 130.0 | 10.8 | 0.8 | 4.7 | 103.4 | 104.0 |
| 135.0 | 8.8 | -3.6 | 2.9 | 108.5 | 109.1 |
| 140.0 | 1.8 | -5.9 | 0.8 | 108.6 | 109.1 |
| 150.0 | -6.7 | -7.6 | -3.7 | 97.5 | 97.6 |
| 160.0 | -6.1 | -9.2 | -7.7 | 79.6 | 79.5 |
| 170.0 | -17.2 | -10.4 | -10.5 | 64.2 | 63.9 |
| 180.0 | -72.8 | -11.3 | -11.5 | 58.3 | 57.9 |

Using the $5^\circ$ equal area mean gravity anomalies estimated from the $1^\circ \times 1^\circ$ anomalies used for the empirical covariance functions given in section 3, and with the procedures described by Rapp (1972) we have computed empirical covariance values using equation (4). The values are shown as plusses in Figure 10. This covariance function can be represented by a spherical cap mean anomaly covariance function with $\psi_0 = 2.^\circ 821$ (cf. section 2). Values are shown in Figure 10 as small circles as computed from equation (11) with the anomaly degree variance model 4 with $\sigma_2^2(\Delta g, \Delta g) = 7.5$ and the summation taken to $n = 144$. For a height of 98.45 km the point anomaly variances becomes equal to the variance of the $5^\circ \times 5^\circ$ equal area mean gravity anomalies, 298.3 $mgal^2$. The graph of this covariance function is shown as a solid line in Figure 10 as well. Again, we can observe a good agreement between the different covariance function.


mgal
400
300
200
100
Figure 10
Graph of point gravity covariance function $cov_M(\Delta g_{Q3}, \Delta g_{Q4})$ for $h_{Q_3} = h_{Q_4} = 98.45$km.
+ Empirical determined $5^\circ \times 5^\circ$ mean gravity anomaly covariance values.
⊖ Spherical cap mean anomaly covariance values for $\psi_0 = 2^\circ.821$.
0°
15°
30°
45°
60°
75°
90°
105°
120°
135°
150°
165°
180°
ψ

## 11. Summary and Conclusion

Least squares collocation is a method of estimating various gravimetric dependent quantities through knowledge of the covariances between such quantities. This report has developed a new model for anomaly degree variances from which covariances for various quantities can be derived with closed formulas. Thus, these covariances between anomalies, height anomalies or (geoid undulations), deflections, etc., are all self-consistent since they are derived from a single starting point, an anomaly degree variance model.

The covariances implied by the results of this report are basically global in nature. This arises from the manner in which the anomaly degree variance model was developed where consideration was given to low degree information concerning the earth's gravitational field, and the global variances of point $1^\circ$ and $5^\circ$ gravity anomalies. It is shown, however, in Section 7 how the global covariance functions can be easily modified to obtain local covariance functions. In addition, mean covariance functions can reasonably be approximated by the point covariance functions evaluated for certain heights above the surface of the earth as explained in Section 9.

Although several anomaly degree variance models and their corresponding covariance functions are discussed, the model recommended was Model 4, defined by equation (25A) and the constants of Table Seven. Numerical results from this model are reported in the text as computed from a computer program utilizing subroutine COVA given as a Fortran program in the appendix. This latter program may be used to evaluate needed covariances to be used in any applications of least squares collocation involving anomalies, height anomalies, and deflections of the vertical.

## References


Clenshaw, C.W., A note on the summation of Chebyshev Series, Math. Tables and other Aids to Computation, Vol. 9, No. 49, p. 118, 1955.

Gradshkyn, I.S., and I.M. Ryzhik, Table of Integrals, Series and Products, 4th edition, Academic Press, New York and London, 1965.

Groten, E., On Linear Regression Prediction of Mean Gravity Anomalies, in Gravity Anomalies: Unsurveyed Areas, American Geophysical Union Monograph, Number 9, H. Orlin editor, 1966.

Heiskanen, W., and H. Moritz, Physical Geodesy, W.H. Freeman, San Francisco, 1967.

Hirvonen, R.A., Statistical Analysis of Gravity Anomalies, Dept. of Geodetic Science Report No. 19, April, 1962, AD275160.

Jordan, S., Self-Consistent Statistical Models for the Gravity Anomaly, Vertical Deflections and Undulations of the Geoid, Journal of Geophysical Research, Vol. 77, No. 20, p. 3660, 1972.

Kaula, W.M., Statistical and Harmonic Analysis of Gravity, Journal of Geophysical Research, Vol. 64, No. 12, 2401-2422, Dec., 1959.

Kaula, W.M., Theory of Statistical Analysis of Data Distributed Over a Sphere in Orbital Perturbations from Terrestrial Gravity Data by W. Kaula, W.H.K. Lee, P.T. Taylor and H.S. Lee, Final Report, Contract AF(601)-4171 (USAF Aeronautical Chart and Information Center, St. Louis), March, 1966a.

Kaula, W.M., Theory of Satellite Geodesy, Blaisdell Publishing Co., 1966b.

Kaula, W.M., Tests and combination of satellite determinations of the gravity field with gravimetry, J. Geophys. Res., 71, 22, pp. 5303-5314, 1966c.

Lauer, S., On the Stochastic Properties of Local Gravity Anomalies and their Prediction (translation of a German dissertation dated, 1971) Defense Mapping Agency Aerospace Center, St. Louis, TC-1873, July, 1973.

Lauritzen, S.L., The Probabilistic Background of Some Statistical Methods in Physical Geodesy, Danish Geodetic Institute, Meddelelse No. 48, 1973.

Meissl, P., A Study of Covariance Functions Related to the Earth's Disturbing Potential, Dept. of Geodetic Science Report No. 151, April, 1971.

Moritz, H., Advanced Least-Squares Methods, Dept. of Geodetic Science Report No. 175, June, 1972.

Pellinen, L.P., A method for expanding the gravity potential of the earth in spherical functions, Trans. Cent. Sci. Res. Inst. Geod. Aerial Surv. Cartogr., no. 171, 1966 (translated by Aeronautical Chart and Information Center, TC-1292, pp. 65-116, 1966, AD 661810).

Rapp, R.H., The formation and analysis of a $5^\circ$ equal area block terrestrial gravity field, Dept. of Geodetic Science, Report No. 178, June, 1972.

Rapp, R.H., Geopotential Coefficient Behavior to High Degree and Geoid Information by Wavelength, Dept. of Geodetic Science, Report No. 180., 1972.

Rapp, R.H., Improved Models for Potential Coefficients and Anomaly Degree Variances, Journal of Geophysical Research, Vol. 78, 17, 3497-3500, June, 1973a.

Rapp, R.H., Current Estimates of Mean Earth Ellipsoid Parameters, Geophysical Research Letters, Vol. 1, No. 1, May, 1974.

Tscherning, C.C., Representation of Covariance Functions Related to the Anomalous Potential of the Earth using Reproducing Kernels, The Danish Geodetic Institute, Internal Report No. 3, Copenhagen, 1972.

## Appendices

Appendix A - Table A - Original 1° Covariance Results

Appendix B - Table B - Anomaly Degree Variances from the Modified 1° Covariance Function

Appendix C - Computer Program for subroutine COVA.

Table A

Original 1° Covariance Results

| Number of Product Pairs | $\psi^\circ$ | $\overline{C}(\psi)$ (mgal)$^2$ | Number of Product Pairs | $\psi^\circ$ | $\overline{C}(\psi)$ (mgal)$^2$ |
|---|---|---|---|---|---|
| 21828. | 0.0 | 996.66 | 1764605.0 | 50.006 | -17.40 |
| 67757.0 | 1.164 | 523.91 | 1786245.0 | 51.005 | -18.07 |
| 109192.0 | 2.101 | 349.25 | 1806095.0 | 52.005 | -19.60 |
| 156505.0 | 3.049 | 285.34 | 1826864.0 | 53.005 | -20.37 |
| 231698.0 | 4.046 | 266.68 | 1844149.0 | 54.004 | -19.37 |
| 255476.0 | 5.060 | 227.34 | 1868090.0 | 55.003 | -19.95 |
| 316123.0 | 6.038 | 212.10 | 1887521.0 | 56.003 | -20.25 |
| 352844.0 | 7.027 | 193.81 | 1914884.0 | 57.005 | -20.02 |
| 410614.0 | 8.022 | 184.37 | 1916381.0 | 58.005 | -18.34 |
| 462226.0 | 9.031 | 179.38 | 1946611.0 | 59.004 | -17.93 |
| 488882.0 | 10.033 | 157.95 | 1961243.0 | 60.005 | -18.71 |
| 541557.0 | 11.022 | 149.53 | 1975934.0 | 61.005 | -19.37 |
| 579519.0 | 12.019 | 130.63 | 1990242.0 | 62.003 | -18.59 |
| 630360.0 | 13.019 | 124.89 | 2009601.0 | 63.003 | -17.64 |
| 664455.0 | 14.019 | 109.62 | 2026129.0 | 64.004 | -18.51 |
| 702812.0 | 15.016 | 96.16 | 2027477.0 | 65.003 | -18.87 |
| 755997.0 | 16.016 | 89.68 | 2050882.0 | 66.002 | -19.63 |
| 787650.0 | 17.021 | 82.13 | 2051483.0 | 67.001 | -19.39 |
| 826509.0 | 18.020 | 74.36 | 2071575.0 | 68.001 | -20.23 |
| 860109.0 | 19.017 | 67.34 | 2079142.0 | 69.002 | -21.42 |
| 903486.0 | 20.015 | 60.76 | 2077609.0 | 70.001 | -21.16 |
| 939201.0 | 21.013 | 54.93 | 2094994.0 | 70.999 | -21.45 |
| 975421.0 | 22.012 | 47.43 | 2105922.0 | 72.001 | -22.20 |
| 1015701.0 | 23.012 | 41.32 | 2101952.0 | 73.003 | -21.10 |
| 1042704.0 | 24.010 | 32.62 | 2105738.0 | 74.002 | -19.75 |
| 1092474.0 | 25.011 | 29.11 | 2102632.0 | 75.001 | -20.06 |
| 1114590.0 | 26.012 | 23.12 | 2116443.0 | 76.000 | -21.50 |
| 1156002.0 | 27.014 | 17.58 | 2110570.0 | 77.000 | -21.97 |
| 1179137.0 | 28.012 | 13.35 | 2115762.0 | 77.999 | -22.09 |
| 1218563.0 | 29.011 | 9.12 | 2117756.0 | 78.999 | -20.67 |
| 1250747.0 | 30.012 | 6.69 | 2113185.0 | 79.999 | -20.75 |
| 1273942.0 | 31.010 | 2.96 | 2117032.0 | 81.000 | -20.98 |
| 1310993.0 | 32.007 | 1.64 | 2105401.0 | 82.000 | -22.20 |
| 1348712.0 | 33.010 | -0.51 | 2103280.0 | 82.998 | -21.60 |
| 1367757.0 | 34.012 | -4.50 | 2107743.0 | 83.997 | -21.02 |
| 1398827.0 | 35.011 | -7.22 | 2102737.0 | 85.000 | -21.26 |
| 1418730.0 | 36.010 | -9.18 | 2087672.0 | 85.999 | -19.37 |
| 1450340.0 | 37.008 | -9.94 | 2084487.0 | 86.998 | -18.26 |
| 1477700.0 | 38.009 | -11.54 | 2085244.0 | 87.999 | -17.64 |
| 1504202.0 | 39.010 | -10.81 | 2071837.0 | 89.001 | -17.47 |
| 1523277.0 | 40.008 | -11.78 | 2045875.0 | 89.998 | -16.56 |
| 1558690.0 | 41.009 | -10.19 | 2055334.0 | 90.995 | -16.23 |
| 1574082.0 | 42.008 | -10.18 | 2047715.0 | 91.998 | -14.73 |
| 1596818.0 | 43.006 | -8.15 | 2033495.0 | 92.998 | -14.07 |
| 1628569.0 | 44.005 | -8.61 | 2015615.0 | 93.997 | -12.31 |
| 1653239.0 | 45.006 | -9.50 | 2013437.0 | 94.996 | -11.32 |
| 1680734.0 | 46.009 | -11.61 | 2004502.0 | 95.998 | -9.30 |
| 1686870.0 | 47.008 | -12.45 | 1980606.0 | 96.999 | -7.56 |
| 1726841.0 | 48.007 | -13.82 | 1968387.0 | 97.997 | -5.72 |
| 1741001.0 | 49.007 | -13.93 | 1961354.0 | 98.996 | -3.04 |

| | | |
|---|---|---|
| 1948486.0 | 99.997 | -0.60 |
| 1934609.0 | 100.998 | 0.99 |
| 1920081.0 | 101.998 | 4.39 |
| 1900164.0 | 102.997 | 4.85 |
| 1893305.0 | 103.997 | 6.68 |
| 1868003.0 | 104.996 | 7.21 |
| 1855030.0 | 105.995 | 6.92 |
| 1840927.0 | 106.994 | 8.65 |
| 1829592.0 | 107.996 | 10.10 |
| 1812532.0 | 108.998 | 12.34 |
| 1785540.0 | 109.997 | 14.40 |
| 1777238.0 | 110.995 | 16.23 |
| 1763249.0 | 111.997 | 18.94 |
| 1733448.0 | 112.997 | 20.09 |
| 1725795.0 | 113.996 | 20.36 |
| 1694670.0 | 114.994 | 22.09 |
| 1689370.0 | 115.993 | 21.49 |
| 1667306.0 | 116.995 | 22.06 |
| 1643495.0 | 117.995 | 21.43 |
| 1625432.0 | 118.993 | 20.82 |
| 1611598.0 | 119.993 | 21.89 |
| 1592001.0 | 120.994 | 23.02 |
| 1562632.0 | 121.993 | 22.46 |
| 1556133.0 | 122.993 | 20.39 |
| 1529283.0 | 123.995 | 19.34 |
| 1507415.0 | 124.995 | 20.00 |
| 1481503.0 | 125.994 | 18.47 |
| 1461365.0 | 126.993 | 18.46 |
| 1437710.0 | 127.993 | 18.96 |
| 1416123.0 | 128.993 | 20.20 |
| 1390757.0 | 129.992 | 21.32 |
| 1365957.0 | 130.991 | 19.60 |
| 1348160.0 | 131.991 | 16.58 |
| 1312392.0 | 132.990 | 14.76 |
| 1299511.0 | 133.990 | 15.17 |
| 1268704.0 | 134.992 | 14.09 |
| 1242453.0 | 135.993 | 12.43 |
| 1210273.0 | 136.992 | 7.48 |
| 1182173.0 | 137.990 | 6.10 |
| 1163148.0 | 138.989 | 4.28 |
| 1128817.0 | 139.990 | 2.79 |
| 1105977.0 | 140.988 | 0.51 |
| 1077246.0 | 141.990 | 0.04 |
| 1048226.0 | 142.991 | -0.47 |
| 1014710.0 | 143.988 | -1.77 |
| 992000.0 | 144.987 | -2.86 |
| 960402.0 | 145.986 | -4.64 |
| 939550.0 | 146.989 | -6.20 |
| 906160.0 | 147.991 | -7.25 |
| 873340.0 | 148.988 | -7.19 |
| 852904.0 | 149.987 | -7.80 |
| 822165.0 | 150.989 | -4.83 |
| 789927.0 | 151.987 | -4.22 |
| 769423.0 | 152.985 | -4.53 |
| 736964.0 | 153.986 | -4.96 |
| 717976.0 | 154.987 | -7.27 |
| 681178.0 | 155.988 | -5.12 |
| 657584.0 | 156.987 | -5.01 |
| 627832.0 | 157.987 | -3.86 |
| 600252.0 | 158.987 | -7.10 |
| 571041.0 | 159.986 | -6.74 |
| 538097.0 | 160.983 | -8.25 |
| 513528.0 | 161.980 | -12.24 |
| 485066.0 | 162.978 | -12.73 |
| 463013.0 | 163.984 | -17.85 |
| 427031.0 | 164.984 | -17.72 |
| 400307.0 | 165.982 | -17.77 |
| 376038.0 | 166.983 | -19.51 |
| 342830.0 | 167.983 | -17.32 |
| 316084.0 | 168.980 | -18.81 |
| 282284.0 | 169.969 | -23.79 |
| 263998.0 | 170.969 | -27.46 |
| 234373.0 | 171.980 | -28.13 |
| 197286.0 | 172.976 | -36.25 |
| 174514.0 | 173.965 | -40.11 |
| 138568.0 | 174.944 | -40.99 |
| 122845.0 | 175.952 | -44.19 |
| 84807.0 | 176.959 | -51.04 |
| 55162.0 | 177.912 | -64.50 |
| 31636.0 | 178.836 | -57.54 |
| 4922.0 | 179.854 | -66.82 |

## Table B
### Anomaly Degree Variances From The Modified 1° Covariance Function

| $\ell$ | $s^{-(\ell+2)}$ | $\beta_\ell$ | $\overline{C}_\ell$† (mgal²) | $C_\ell$* (mgal²) |
|---|---|---|---|---|
| 0 | 1.00077 | 1.00000 | 0.07 | 0.07 |
| 1 | 1.00115 | 0.99998 | 0.02 | 0.02 |
| 2 | 1.00153 | 0.99993 | 7.54 | 7.56 |
| 3 | 1.00192 | 0.99985 | 33.88 | 33.95 |
| 4 | 1.00230 | 0.99976 | 19.17 | 19.23 |
| 5 | 1.00269 | 0.99964 | 21.57 | 21.64 |
| 6 | 1.00307 | 0.99949 | 18.87 | 18.95 |
| 7 | 1.00345 | 0.99932 | 18.77 | 18.86 |
| 8 | 1.00384 | 0.99913 | 10.42 | 10.48 |
| 9 | 1.00422 | 0.99891 | 11.05 | 11.12 |
| 10 | 1.00461 | 0.99867 | 11.43 | 11.51 |
| 11 | 1.00499 | 0.99840 | 14.10 | 14.22 |
| 12 | 1.00538 | 0.99811 | 3.12 | 3.15 |
| 13 | 1.00576 | 0.99780 | 9.47 | 9.56 |
| 14 | 1.00615 | 0.99746 | 5.76 | 5.82 |
| 15 | 1.00653 | 0.99710 | 7.58 | 7.67 |
| 16 | 1.00692 | 0.99671 | 9.93 | 10.07 |
| 17 | 1.00730 | 0.99630 | 8.63 | 8.76 |
| 18 | 1.00769 | 0.99586 | 8.26 | 8.40 |
| 19 | 1.00808 | 0.99540 | 7.67 | 7.80 |
| 20 | 1.00846 | 0.99492 | 1.16 | 1.18 |
| 21 | 1.00885 | 0.99441 | 5.81 | 5.92 |
| 22 | 1.00924 | 0.99388 | 4.47 | 4.56 |
| 23 | 1.00962 | 0.99333 | 5.93 | 6.06 |
| 24 | 1.01001 | 0.99275 | 6.19 | 6.34 |
| 25 | 1.01040 | 0.99215 | 9.24 | 9.49 |
| 26 | 1.01078 | 0.99152 | 1.96 | 2.02 |
| 27 | 1.01117 | 0.99087 | 4.73 | 4.87 |
| 28 | 1.01156 | 0.99020 | 4.17 | 4.30 |
| 29 | 1.01195 | 0.98950 | 4.98 | 5.15 |
| 30 | 1.01233 | 0.98878 | 3.89 | 4.02 |
| 31 | 1.01272 | 0.98803 | 4.82 | 5.00 |
| 32 | 1.01311 | 0.98726 | 7.78 | 8.09 |
| 33 | 1.01350 | 0.98647 | 6.90 | 7.19 |
| 34 | 1.01389 | 0.98566 | 5.89 | 6.15 |
| 35 | 1.01427 | 0.98482 | 7.63 | 7.98 |
| 36 | 1.01466 | 0.98395 | 6.42 | 6.73 |
| 37 | 1.01505 | 0.98307 | 4.56 | 4.79 |
| 38 | 1.01544 | 0.98216 | 7.39 | 7.77 |
| 39 | 1.01583 | 0.98122 | 5.64 | 5.95 |
| 40 | 1.01622 | 0.98027 | 5.41 | 5.72 |
| 41 | 1.01661 | 0.97929 | 5.45 | 5.78 |
| 42 | 1.01700 | 0.97828 | 6.46 | 6.87 |
| 43 | 1.01739 | 0.97726 | 5.01 | 5.34 |
| 44 | 1.01778 | 0.97621 | 5.56 | 5.93 |
| 45 | 1.01817 | 0.97514 | 7.54 | 8.08 |
| 46 | 1.01856 | 0.97404 | 2.81 | 3.02 |
| 47 | 1.01895 | 0.97292 | 5.78 | 6.22 |
| 48 | 1.01934 | 0.97178 | 3.64 | 3.93 |
| 49 | 1.01973 | 0.97062 | 5.51 | 5.96 |
| 50 | 1.02012 | 0.96943 | 5.61 | 6.09 |
| 51 | 1.02051 | 0.96822 | 3.48 | 3.79 |
| 52 | 1.02090 | 0.96699 | 4.03 | 4.40 |
| 53 | 1.02129 | 0.96573 | 5.72 | 6.27 |
| 54 | 1.02168 | 0.96446 | 4.00 | 4.39 |
| 55 | 1.02208 | 0.96316 | 4.65 | 5.12 |
| 56 | 1.02247 | 0.96183 | 4.15 | 4.59 |
| 57 | 1.02286 | 0.96049 | 4.34 | 4.81 |
| 58 | 1.02325 | 0.95912 | 5.11 | 5.68 |
| 59 | 1.02364 | 0.95773 | 5.46 | 6.09 |
| 60 | 1.02403 | 0.95632 | 2.88 | 3.23 |
| 61 | 1.02443 | 0.95489 | 3.94 | 4.42 |
| 62 | 1.02482 | 0.95343 | 3.83 | 4.32 |
| 63 | 1.02521 | 0.95195 | 3.83 | 4.34 |
| 64 | 1.02561 | 0.95045 | 4.14 | 4.69 |
| 65 | 1.02600 | 0.94893 | 2.51 | 2.86 |
| 66 | 1.02639 | 0.94739 | 5.02 | 5.74 |
| 67 | 1.02678 | 0.94582 | 4.66 | 5.35 |
| 68 | 1.02718 | 0.94424 | 4.75 | 5.48 |
| 69 | 1.02757 | 0.94263 | 4.91 | 5.68 |
| 70 | 1.02797 | 0.94100 | 2.11 | 2.45 |
| 71 | 1.02836 | 0.93935 | 3.22 | 3.75 |
| 72 | 1.02875 | 0.93767 | 3.98 | 4.66 |
| 73 | 1.02915 | 0.93598 | 4.01 | 4.71 |
| 74 | 1.02954 | 0.93427 | 4.72 | 5.57 |
| 75 | 1.02994 | 0.93253 | 4.37 | 5.18 |
| 76 | 1.03033 | 0.93077 | 2.77 | 3.29 |
| 77 | 1.03073 | 0.92900 | 4.88 | 5.83 |
| 78 | 1.03112 | 0.92720 | 4.12 | 4.94 |
| 79 | 1.03152 | 0.92538 | 3.01 | 3.63 |
| 80 | 1.03191 | 0.92354 | 6.36 | 7.70 |
| 81 | 1.03231 | 0.92168 | 3.93 | 4.77 |
| 82 | 1.03270 | 0.91980 | 5.17 | 6.31 |
| 83 | 1.03310 | 0.91790 | 5.42 | 6.65 |
| 84 | 1.03349 | 0.91598 | 3.44 | 4.24 |
| 85 | 1.03389 | 0.91403 | 4.80 | 5.94 |
| 86 | 1.03429 | 0.91207 | 6.24 | 7.76 |
| 87 | 1.03468 | 0.91009 | 5.12 | 6.39 |
| 88 | 1.03508 | 0.90809 | 4.80 | 6.03 |
| 89 | 1.03547 | 0.90607 | 3.68 | 4.65 |
| 90 | 1.03587 | 0.90403 | 4.63 | 5.87 |
| 91 | 1.03627 | 0.90197 | 4.18 | 5.32 |
| 92 | 1.03667 | 0.89989 | 4.95 | 6.34 |
| 93 | 1.03706 | 0.89779 | 2.92 | 3.76 |
| 94 | 1.03746 | 0.89567 | 3.39 | 4.38 |
| 95 | 1.03786 | 0.89353 | 2.06 | 2.67 |
| 96 | 1.03825 | 0.89137 | 4.17 | 5.44 |
| 97 | 1.03865 | 0.88920 | 2.56 | 3.36 |
| 98 | 1.03905 | 0.88700 | 4.72 | 6.24 |
| 99 | 1.03945 | 0.88479 | 2.78 | 3.69 |

* from equation (16A)



† from equation (16) with s = 1

| | | | | |
|---|---|---|---|---|
| 100 | 1.03985 | 0.88255 | 3.81 | 5.09 |
| 101 | 1.04025 | 0.88030 | 2.58 | 3.46 |
| 102 | 1.04064 | 0.87803 | 3.31 | 4.47 |
| 103 | 1.04104 | 0.87575 | 2.86 | 3.89 |
| 104 | 1.04144 | 0.87344 | 3.62 | 4.95 |
| 105 | 1.04184 | 0.87111 | 2.00 | 2.74 |
| 106 | 1.04224 | 0.86877 | 3.47 | 4.79 |
| 107 | 1.04264 | 0.86641 | 3.08 | 4.27 |
| 108 | 1.04304 | 0.86403 | 3.27 | 4.58 |
| 109 | 1.04344 | 0.86164 | 3.19 | 4.48 |
| 110 | 1.04384 | 0.85922 | 3.27 | 4.62 |
| 111 | 1.04424 | 0.85679 | 3.06 | 4.35 |
| 112 | 1.04464 | 0.85434 | 4.21 | 6.02 |
| 113 | 1.04504 | 0.85188 | 3.55 | 5.12 |
| 114 | 1.04544 | 0.84939 | 2.47 | 3.58 |
| 115 | 1.04584 | 0.84690 | 2.31 | 3.37 |
| 116 | 1.04624 | 0.84438 | 3.04 | 4.46 |
| 117 | 1.04664 | 0.84184 | 2.78 | 4.10 |
| 118 | 1.04704 | 0.83929 | 2.16 | 3.21 |
| 119 | 1.04744 | 0.83673 | 2.91 | 4.36 |
| 120 | 1.04784 | 0.83414 | 1.76 | 2.66 |
| 121 | 1.04825 | 0.83154 | 2.73 | 4.13 |
| 122 | 1.04865 | 0.82893 | 1.95 | 2.98 |
| 123 | 1.04905 | 0.82630 | 1.62 | 2.49 |
| 124 | 1.04945 | 0.82365 | 2.59 | 4.00 |
| 125 | 1.04985 | 0.82099 | 1.48 | 2.31 |
| 126 | 1.05026 | 0.81831 | 2.73 | 4.28 |
| 127 | 1.05066 | 0.81561 | 1.92 | 3.03 |
| 128 | 1.05106 | 0.81290 | 2.94 | 4.68 |
| 129 | 1.05146 | 0.81017 | 1.22 | 1.96 |
| 130 | 1.05187 | 0.80743 | 2.70 | 4.36 |
| 131 | 1.05227 | 0.80468 | 1.75 | 2.84 |
| 132 | 1.05267 | 0.80191 | 2.98 | 4.87 |
| 133 | 1.05308 | 0.79912 | 1.90 | 3.13 |
| 134 | 1.05348 | 0.79632 | 2.79 | 4.63 |
| 135 | 1.05388 | 0.79351 | 1.62 | 2.71 |
| 136 | 1.05429 | 0.79068 | 1.05 | 1.77 |
| 137 | 1.05469 | 0.78783 | 1.16 | 1.98 |
| 138 | 1.05509 | 0.78498 | 2.49 | 4.26 |
| 139 | 1.05550 | 0.78210 | 0.90 | 1.55 |
| 140 | 1.05590 | 0.77922 | 1.15 | 2.00 |

## Appendix - Subroutine COVA

A subroutine COVA for the computation of the covariance of and between height anomalies, gravity anomalies and the longitudinal and transveral components of the deflections of the vertical is reproduced below.

The FORTRAN IV language of the IBM 360/370 system has been used.

The subroutine can only be used for the computation of covariances corresponding to the degree-variance model given by equation (68).

By the execution of a DATA statement, the quantities s, A and B become equal to the values given in Table Seven. It is only necessary to change the values given in the DATA statement to obtain the covariances corresponding to a degree-variance model with other values of s, A and B. The subroutine can be used to compute covariance values corresponding to both a local n'th order covariance function and to a covariance function, which has some of the degree-variances equal to empirical determined values.

The comments given in connection with the Fortran statements of the subroutine should give all details necessary for the application of the subroutine.

```
      SUBROUTINE COVA(EPS,N1)
C
C  THE SUBROUTINE COMPUTES ONE OF SEVEN DIFFERENT COVARIANCES (SEE BE-
C  LOW), USING THE ANOMALY DEGREE-VARIANCE MODEL GIVEN THROUGH THE VAL-
C  UES OF TABLE SEVEN AND EQUATION (68).(THE QUANTITY S IN THE TABLE IS
C  HERE CALLED SE).
C  THERE ARE THREE ENTRIES TO THE SUBROUTINE, WHICH HAVE TO BE CALLED IN
C  THE SEQUENCE COVA,COVB AND COVC.
C
C  BY THE CALL OF COVA, THE KIND OF COVARIANCE FUNCTION TO BE USED IS
C  DETERMINED. THERE ARE THREE POSSIBILITIES:
C  (1) THE COVARIANCE MODEL FOUR (EQUATIONS (130)-(132) AND (136)-(139))
C      IS USED WITHOUT MODIFICATIONS. IN THIS CASE EPS WILL BE A DUMMY
C      ARRAY AND N1 MUST BE EQUAL TO ONE.
C      THE LOGICAL VARIABLE MODEL WILL GET THE VALUE TRUE IN THIS CASE.
C  (2) A NUMBER (N1) OF THE ANOMALY DEGREE-VARIANCES (DEGREE ZERO TO
C      N1-1) ARE PUT EQUAL TO EMPIRICAL DETERMINED DEGREE-VARIANCES.
C      THE DEGREE-VARIANCE OF DEGREE K WILL HAVE TO BE STORED IN
C      EPS(K+1) (IN UNITS OF MGAL**2).
C  (3) THE DEGREE-VARIANCES OF DEGREE ZERO TO N = N1-1 ARE PUT EQUAL TO
C      ZERO,(AND THE OTHERS ARE THE SAME AS ABOVE DESCRIBED). THIS MEANS
C      THAT AN N'TH ORDER LOCAL COVARIANCE FUNCTION WILL BE COMPUTED. IN
C      THIS CASE EPS MUST HAVE N1 ZERO VALUES STORED.
C  IN ALL CASES N1 MUST BE LESS THAN 300 AND EPS MUST HAVE DIMENSION
C  N1.
      IMPLICIT REAL *8(A-H,O-Z)
      LOGICAL MODEL,NOTD,NOTDD
      DIMENSION EPSC(300),EPS(1)
      DATA RE,GM,A,SE,B,IB1,IB2,IBM1,EPSC(1),EPSC(2),D0,D1,D2,D3,D4,
     *D5,RADSEC/6371.0D3,3.98D14,425.28D0,0.999617D0,24.0D0,25,26,23,
     *3*0.0D0,1.0D0,2.0D0,3.0D0,4.0D0,1.0D5,206264.806D0/
      IB12 = IB1*IB2
      RADSE2 = RADSEC**2
      RE2 = RE*RE
      RBJ2 = RE2*SE
      RBJ = DSQRT(RBJ2)
      AM = A/D5
      AM2 = AM/D5
C  A IS IN UNITS OF MGAL**2, AM IN UNITS OF MGAL*M/SEC AND AM2 IN
C  UNITS OF (M/SEC)**2. RBJ IS THE RADIUS OF THE BJERHAMMAR-SPHERE.
      MODEL = N1.EQ.1
      IF (MODEL) GO TO 20
C
C  WE WILL NOW COMPUTE THE MODIFIED (POTENTIAL) DEGREE-VARIANCES, CF.
C  EQUATION (151).
      IF (N1.LT.3) GO TO 20
      DO 10 I = 3, N1
      RI = DFLOAT(I-1)
      IF (I.EQ.3) EPS(3) = EPS(3)*RBJ2*1.0D-10
   10 IF (I.GT.3) EPS(I) =
     *    RBJ2*(EPS(I)/((RI-D1)**2)*1.0D-10-AM2/((RI-D1)*(RI-D2)*(RI+B)))
   20 RETURN
```

```
      ENTRY COVB(KTYPE)
C   BY THE CALL OF COVB, THE TYPE OF COVARIANCE TO BE COMPUTED IS DETER-
C   MINED BY THE VALUE OF KTYPE, SO THAT WE GET THE COVARIANCE BETWEEN:
C   THE GRAVITY ANOMALY AT P AND THE GRAVITY ANOMALY AT Q     FOR KTYPE=1,
C   THE     -          -     -  -  -  THE LONGITUDIONAL COMPONENT
C                          OF THE DEFLECTION OF THE VERTICAL AT Q FOR KTYPE=2,
C   THE     -          -      AT P AND THE HEIGHT ANOMALY AT Q      FOR KTYPE=3,
C   THE LONGITUDIONAL COMPONENT OF THE DEFLECTION OF THE VERTI-
C                   CAL AT P AND THE SAME TYPE OF QUANTITY AT Q FOR KTYPE=4,
C   THE TRANSVERSAL COMPONENT OF THE DEFLECTION OF THE VERTI-
C                   CAL AT P AND THE SAME TYPE OF QUANTITY AT Q FOR KTYPE=5,
C   THE LONGITUDIONAL COMPONENT OF THE DEFLECTION OF THE VERTI-
C                         CAL AT P AND THE HEIGHT ANOMALY AT Q  FOR KTYPE=6,
C   AND THE HEIGHT ANOMALY AT P AND THE HEIGHT ANOMALY AT Q  FOR KTYPE=7.
C
C   THE VALUE OF KTYPE WILL THEN ALSO DETERMINE WHICH OF THE COEFFICIENTS
C   (151)-(153),THAT WE WILL USE IN THE EVALUATION OF THE LEGENDRE-SERIES
C   AND Whether NO DIFFERENTIATION, DIFFERENTIATION ONE TIME OR DIFFEREN-
C   TIATION TWO TIMES WITH RESPECT TO THE VARIABLE T TAKES PLACE. TWO
C   LOGICAL VARIABLES NOTD AND NOTDD ARE USED TO DISTINGUISH BETWEEN THE
C   SITUATIONS.
      IF (MODEL) GO TO 35
C
      IF (KTYPE.EQ.1) IP = 2
      IF (KTYPE.EQ.2.OR.KTYPE.EQ.3) IP = 1
      IF (KTYPE.GT.3) IP = 0
      DO 30 I = 3, N1
   30 EPSC(I) = EPS(I)*((I-2)*D5/RBJ)**IP
C
   35 NOTD = KTYPE.EQ.1.OR.KTYPE.EQ.3.OR.KTYPE.EQ.7
      NOTDD = KTYPE.NE.5.AND.KTYPE.NE.4
      RETURN
C
      ENTRY COVC(PSI,HP,HQ,COV)
C   BY THE CALL OF COVC THE COVARIANCE OF TYPE KTYPE WILL BE COMPUTED FOR
C   POINTS P AND Q HAVING SPHERICAL DISTANCE (RADIANS) PSI, WHERE HP IS
C   THE HEIGHT OF P ABOVE THE EARTH AND HQ THE HEIGHT OF Q ABOVE THE
C   EARTH. THE COVARIANCE WILL BE RETURNED BY THE VARIABLE COV. UNITS ARE
C   PRODUCTS OF MGAL, METERS AND ARCSECONDS.
C
      T = DCOS(PSI)
      U = DSIN(PSI)
      T2 = T*T
      U2 = U*U
      RP = RE+HP
      RQ = RE+HQ
      S = RBJ2/(RP*RQ)
      S2 = S*S
      S3 = S2*S
      TS = T*S
      P2 = (D3*T2-D1)/D2
      GP = GM/(RP*RP)
      GQ = GM/(RQ*RQ)
```

```
C   THE QUANTITIES L,M AND N  DEFINED IN EQ.(75) ARE HERE CALLED SL,SM
C   AND SN. L**2 = SL2.
       SL2 = D1+S2-D2*TS
       SL = DSQRT(SL2)
       SL3 = SL2*SL
       SN = D1-TS+SL
       SM = D1-TS-SL
       SLN = SL*SN
       SLNL = -DLOG(SN/D2)
C
C   WHEN WE ARE COMPUTING A LOCAL N'TH ORDER COVARIANCE OR A COVARIANCE
C   FROM A GLOBAL MODEL WITH EMPIRICAL DEGREE-VARIANCES UP TO AND INCLU-
C   SIVE DEGREE N, WE WILL HAVE TO COMPUTE THE SUM  (154), THE SUM (155)
C   (WHEN NOTD IS FALSE) AND THE SUM (156) (WHEN NOTDD IS FALSE). (154)
C   WILL BE ACCUMMULATED IN B0, (155) IN DB0 AND (156) IN DDB0.
C   WHEN THE VARIABLE MODEL IS TRUE, B0, DB0 AND DDB0 WILL BE PUT EQUAL
C   TO ZERO.
C
       B0 = D0
       DB0 = D0
       DDB0 = D0
       IF (MODEL) GO TO 45
C
       B1 = D0
       DB1 = D0
       DDB1 = D0
       L1 = N1
       RL1= DFLOAT(L1)
C
C   WE WILL NOW USE THE RECURSION FORMULAE (183),(185) AND (186), WHERE
C   THE TERM (176A) DIVIDED BY T IS CALLED EL AND FL1 IS THE TERM (176B)
C   FOR SUBSCRIPT L+1.
       DO 40 I = 1, N1
       EL = (D2*RL1-D1)*S/RL1
       FL1 = -RL1*S2/(RL1+D1)
       RL1 = RL1-D1
       B2 = B1
       B1 = B0
       B0 = B1*EL*T+B2*FL1+EPSC(L1)
       IF (NOTD) GO TO 40
C
       DB2 = DB1
       DB1 = DB0
       DB0 = EL*(DB1*T+B1)+FL1*DB2
       IF (NOTDD) GO TO 40
C
       DDB2 = DDB1
       DDB1 = DDB0
       DDB0 = EL*(DB1*D2+DDB1*T)+FL1*DDB2
    40 L1 = L1-1
C
```

```
C   COMPUTATION OF CLOSED EXPRESSIONS. FIRST SOME AUXILLIARY QUANTITIES.
C   FM1 IS THE QUANTITY (86), FM2 IS (87), F1 IS (99) AND F2 IS (100)
   45 DPL = D1+SL
      DML = D1-SL
      P31 = D3*TS+D1
      BO = BO*S
      FM1 = S*(SM+TS*SLNL)
      FM2 = S*(SM*P31/D2+S2*(P2*SLNL+U2/D4))
      F1 = DLOG(D1+D2*S/(D1-S+SL))
      F2 = (SL-D1+T*F1)/S
      IF (NOTD) GO TO 48
C
      DBO = DBO*S
C   DFM1 IS THE QUANTITY (90), DFM2 IS (92), DF1 IS (101) AND DF2 IS
C   (103).
      DFM1 = S2*(DML/SL+SLNL+TS*(D1/SLN+D1/SN))
      DFM2 = S2*((P31/SL+D2-7.0D0*TS-D3*SL)/D2+S*(D3*T*SLNL
     *        +S*P2*DPL/SLN))
      DF1 = S2/SLN
      DF2 = -D1/SL+TS/SLN+F1/S
      DL =-S/SL
      IF (NOTDD) GO TO 48
C
      DDBO = DDBO*S
C   DDFM1 IS THE QUANTITY (91), DDFM2 IS (93), DDF1 IS (102) AND DDF2 IS
C   (104).
      DDFM1 = S3*(D1/SL3+D2*DPL/SLN+TS*(D1/(SL3*SN)+(DPL/SLN)**2))
C
      DDFM2 = S3*((6.0D0/SL+P31/SL3-7.0D0)/D2+D3*SLNL+6.0D0*TS*DPL/SLN
     *        +P2*S2*((DPL/SLN)**2+D1/(SL3*SN)))
      DDF1 = S3*(DPL/SLN**2+D1/(SN*SL3))
      DDF2 = (-S2/SL3+D2*DF1+T*DDF1)/S
      DDL = -S2/SL3
C   WE CAN NOW USE THE RECURSION FORMULAE (96), (97) AND (98) FOR THE
C   COMPUTATION OF THE QUANTITY (73) CALLED FB AND ITS DERIVATIVES DFB
C   AND DDFB.
C
   48 DO 50 I = 2, IBM1
      RI = DFLOAT(I)
      DI2 = D2*RI-D1
      DI1 = (RI-D1)/S
      FB = (SL+DI2*T*F2-DI1*F1)/(RI*S)
      F1 = F2
      F2 = FB
      IF (NOTD) GO TO 50
      DFB = (DL+DI2*(F1+T*DF2)-DI1*DF1)/(RI*S)
      DF1 = DF2
      DF2 = DFB
      IF (NOTDD) GO TO 50
      DDFB = (DDL+DI2*(D2*DF1+T*DDF2)-DI1*DDF1)/(RI*S)
      DDF1 = DDF2
      DDF2 = DDFB
   50 CONTINUE
```

```
      IF (NOTD.OR.KTYPE.EQ.2) GO TO 60
C   FROM (133) WE HAVE:
      DK = DB0+AM2*RBJ2*(IB1*DFM2-IB2*(DFM1-D3*T*S3)+DFB-S2/IB1-D3*S3*T/
     *     IB2)/IB12
   60 GO TO (61,62,63,64,65,66,67),KTYPE
C   EQUATION (132) AND (146) GIVES:
   61 COV = S*B0+A*S*(IB1*(FB-S/B-S2*T/IB1-S3*P2/IB2)+FM2)/IB2
      GO TO 70
C   EQUATION (139) AND (150) GIVES:
   62 COV = U*(DB0*RBJ/(RP*RQ)+AM*S*(DFM2-DFB+S2/IB1+D3*S3*T/IB2)/IB2)/
     *      GQ*RADSEC
      GO TO 70
C   EQUATION (131) AND (145) GIVES:
   63 COV = (B0*RBJ+AM*RBJ2*(FM2-FB+S/B+S2*T/IB1+S3*P2/IB2)/IB2)/
     *(RP*GQ)
      GO TO 70
C   EQUATION (136) AND (147) GIVES:
   64 COV = (T*DK/(RP*RQ)-U2*(DDB0/(RP*RQ)+AM2*S*(IB1*DDFM2-IB2*(DDFM1
     *    -D3*S3)+DDFB-D3*S3/IB2)/IB12))*RADSE2/(GP*GQ)
      GO TO 70
C   EQUATION (137) AND (148) GIVES:
   65 COV = DK/(RP*RQ*GP*GQ)*RADSE2
      GO TO 70
C   EQUATION (138) AND (149) GIVES:
   66 COV = U*DK/(GP*GQ*RP)*RADSEC
      GO TO 70
C AND EQUATION (37), (130) AND (144) GIVES:
   67 COV = (B0+AM2*RBJ2*(IB1*FM2-IB2*(FM1-S3*P2)+FB-S/B-S2*T/IB1-S3*P2
     *     /IB2)/IB12)/(GP*GQ)
   70 RETURN
      END
```