Reports of the Department of Geodetic Science

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COVARIANCE EXPRESSIONS FOR SECOND AND LOWER ORDER DERIVATIVES OF THE ANOMALOUS POTENTIAL

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C. C. Tscherning

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Foreword

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Abstract

Auto- and cross-covariance expressions for the anomalous potential of the Earth and its first and second order derivatives are derived based on three different degree-variance models.

A FORTRAN IV subroutine is listed and documented that may be used for the computation of auto- and cross-covariance between any of the following quantities: (1) the anomalous potential (T), (2) the negative gravity disturbance/r, (3) the gravity anomaly (Ag), (4) the radial component of the gradient of Ag, (5) the second order radial derivative of T, (6), (7) the latitude and longitude components of the deflection of the vertical, (8), (9) the derivatives in northern and eastern direction of Ag, (10), (11) the derivatives of the gravity disturbance in northern and eastern direction, (12) - (14) the second order derivatives of T in northern, in mixed northern and eastern and in eastern direction.

Values of different kinds of covariance of second order derivatives for varying spherical distance and height are tabulated.

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1. Introduction

In this report we will derive expressions for the covariance functions of some quantities related to the anomalous potential of the Earth (denoted T). The quantities which we will consider are values of linear functionals applied to the anomalous potential, e.g. the normal derivative at a specific point on the surface of the Earth.

Covariance expressions for such quantities may be derived from one single covariance function, namely the covariance function of the anomalous potential

$$\mathrm{cov}(T_P, T_Q).$$

This function yields for two points, P and Q, outside the Earth the covariance between the values of the anomalous potential at these two points. It may then as well be regarded as a function of P or Q. The linear functionals, which correspond to quantities between which we want to obtain the covariance, can then be applied to this function. This operation will furnish us with the covariance between the quantities.

Covariance functions of quantities, which included first order derivatives of T have earlier been derived (e.g. Tscherning and Rapp (1974)). These covariance functions were required, e.g. when combining gravity anomalies and deflections of the vertical for the determination of approximations to T using the method of least squares collocation, see Tscherning (1974).

It has now become possible to measure second order derivatives of T in an aircraft. This fact makes these quantities more applicable for several geodetic purposes than the similar quantities measured at the surface of the Earth because topographic reductions are not needed.

The method of least squares collocation should, as advocated by Moritz (1974), be well suited for the filtering of the measurements and for their use in combination with other data for the determination of T. But the knowledge of the covariance expressions for these quantities and between these quantities and other kinds of data is required.

Unfortunately the estimation of the basic covariance function, cov (T_P, T_Q) is difficult both in theory and practice. In Tschernmg and Rapp (1974) the covariance function is chosen as the function which in between a set of models fits the available data the best possible. Another criterion for the choice could have been the usefulness of the covariance function when applied in least squares collocation, i.e. the numerical properties of the covariance function.

These properties do not stand out clearly in the statistical model for least squares collocation. But we may as well regard the method as a functional-analytic approximation method (see Krarup (1969)). Here the choice of the covariance function is clearly seen as being equivalent to the choice of a norm or inner product in a function space. The approximation, which is determined, will fulfill a minimum condition. It will have the least norm between the functions, which agree with the given (filtered) measurements.

The norms, which correspond to the covariance models discussed in Tscherning and Rapp (1974), correspond to inner products, which include derivatives of different order, (e.g. Tscherning (1973)). Therefore, the use of some of these different covariance functions will have the effect, that the approximations will fulfill different minimum conditions, i.e. be smooth in different ways. In this report we will therefore not recommend a specific model, but develop covariance expressions based on models with different numerical characteristics. The final choice of model should then wait until the corresponding covariance functions have been used in numerical tests.

The covariance function of the anomalous potential is here chosen to be rotational!: invariant, and it will be harmonic in each variable P and Q. It may then be expressed as the sum of a Legendre series:

$$\mathrm{cov}(T_P, T_Q) = \sum_{\ell=0}^{\infty} \sigma_\ell(T,T) \left(\frac{R_b^2}{r\, r'}\right)^{\ell+1} P_\ell(\cos\psi)$$

where $\sigma_\ell(T,T)$ are so-called potential degree-variances, $P_\ell(\cos\psi)$ is the ℓ'th order Legendre polynominal, R_b is the radius of a sphere totally enclosed in the Earth, r and r' are the distance of P, Q respectively from the origin and ψ is the spherical distance between P and Q.

The different .numerical properties are reflected in the behavior of the degree-variance for ℓ going to infinity, (e.g. Tscherning (1973)). We will here consider three types of degree-variances, namely $\sigma_a(T,T)$ decreasing towards zero like $1/.Q^2\}/\ell^3$ and $1/\ell^4$, (c. f. eq. (17)).

Let the point P have spherical coordinates (φ, h, r)= (latitude, longitude, distance 'from the origin) and let us denote partial differentiation with respect to a variable, e. g. r, by a capital D having the variable as subscript: Q.

We may then express the quantities between which we have chosen to develop covariance expressions by the following (linear) functionals applied on T:

(1)	$\zeta = T(P)/\gamma$,	the height anomaly, equal to the value of the evaluation functional applied on T,(=T(P)) and divided by the reference gravity,
(2)	$-D_r T \cdot \frac{1}{F}$,	the radial derivative (divided by r),
(3)	$\Delta g = -QT - \frac{2}{r} T(P)$,	the (free air) gravity anomaly,
(4)	$D_r(\Delta g)$,	the gravity gradient,

(5) $D_r^2 T$, the second order radial derivative,

(6) $\xi = -D_\varphi T/(r \cdot \gamma)$, the latitude component of the deflection of the vertical,

(7) $\eta = -D_h T/(r \cdot \gamma \cdot \cos\varphi)$, the longitude component of the deflection of the vertical,

(8) $-(D_\varphi \Delta g)/r$, the derivative of Δg with respect to φ,

(9) $-(D_\lambda \Delta g)/(r \cdot \cos\varphi)$, the derivative of Δg with respect to h,

(10) $(D_\varphi D_r T)/r$

(11) $(D_\lambda D_r T)/(r \cdot \cos\varphi)$,

(12) $(D_\varphi^2 T)/r^2$,

(13) $(D_\varphi D_\lambda T)/(r^2 \cos\varphi)$,

(14) $(D_\lambda^2 T)/(r^2 \cos\varphi^2)$,

From the covariances between these quantities, it is possible to compute covariances of all other first or second order derivatives of T. The covariance of the value of the Laplace operator at P with some other quantity at Q, for example, may be computed using the covariances between the quantities given in equations (2), (5), (8), (12) and (14), because

$$(15) \quad \Delta T = D_r^2 T + \frac{2}{r} D_r T + \frac{1}{r^2} D_\varphi^2 T - \frac{\tan\varphi}{r} D_\varphi T + \frac{1}{r^2 \cos^2\varphi} D_\lambda^2 T.$$

(The covariance between two quantities, where one of these is the quantity $\Delta T = 0$, will naturally always be zero. This fact can be used for the numerical checking of the computations).

The technique used for the derivation of the covariance expressions is quite simple, and follows the technique presented in Tscherning and Rapp (1974) closely. Though, we have chosen to derive equations as general as possible. This appeared to be an advantage, when the equations were programmed in FORTRAN IV.

The point of departure is an expression for $\text{cov}(T_P, T_Q)$. The expressions for other quantities are then found by applying the linear functionals related to these quantities on this expression.

The function $\text{cov}(T_P, T_Q)$ is given by the choice of the radius R_b of a Bjerhammar-sphere and of a set of potential degree-variances $\sigma_\ell(T,T)$, (cf. Tscherning and Rapp

(1974, page 2)). We will, as in that publication, prescribe a rule for the general behavior of the degree-variances, by adopting a certain degree-variance model. For the first n (where n is an integer ≥ 0) degree-variances a correction to the model degree-variance is allowed, so that, for example, the model degree-variance plus the correction is equal to an empirical determined degree-variance:

$$(16) \qquad \operatorname{cov}(T_P T_Q) = \sum_{\ell=0}^{n} \sigma_\ell^c (T,T)\, s^{\ell+1} P_\ell(t)$$

$$+ \sum_{\ell=0}^{\infty} \sigma_\ell^m (T,T)\, s^{\ell+1} P_\ell(t),$$

where the superscripts c and m stands for "correction" and "model" respectively, $s = R_b^2/(r \cdot r')$ and $t = \cos\psi$.

The finite sum can be computed using a simple recursion algorithm and the infinite sum can (depending on the choice of model) be evaluated using a corresponding analytic expresssion.

The application of the functionals (1)- (14) may be divided in two steps, namely by first performing the necessary differentiations with respect to r and r' and then with respect to the other spherical coordinates. The two following sections will deal with this problem. In section 4 we will discuss a FORTAN IV subroutine, which may be used for computations and we will finally in section 5 present tables of covariances and discuss methods for the evaluation of the reliability of the computed quantities. An appendix contain the listing of the subroutine and an example of a FORTRAN IV program using the subroutine.

2. The second and lower order radial derivatives of $\operatorname{cov}(T_P, T_Q)$.

We will here regard model (potential) degree-variances of 'the form

$$(17) \qquad \sigma_{\ell,i}^m = A_i \prod_{j=0}^{i} (\ell + k_j)^{-1},$$

where the superscript i (=1, 2 or 3) is a "model" number. A_i is a constant in units of $(m/sec)^4$.

The quantities k_j will all be integers greater than an integer p. The degree-variances of degree less than -p will have to be zero, when p is negative. For example, let the smallest k_j be equal to -2. Then p will be equal to -3, and the model degree-variances of degree 0, 1 and 2 will be equal to zero.

(Note; that the models 1, 2, 3 regarded here correspond to Tscherning and Rapp (1974, p. 30) model 3, 4, 5 with $k_0 = -2$ and $k_1 = -1$ for model 1, $k_0 = -2$, $k_1 = -1$, $k_2 = i$ for model 2 and $k_0 = -2$, $k_1 = -1$, $k_2 = i$ and $k_3 = j$ for model 3 and $A = A_i * 10^{10}/R_b^2$ for R_b given in meters).

The degree-variances may be expressed by a sum of partial fractions

(18) $$\sigma^{m}_{\ell,i} = A_i \sum_{j=0}^{i} c_j/(\ell + k_j),$$

where

(19) $$c_j = \prod_{m=0,\, m\neq j}^{i} 1/(k_m - k_j).$$

(This may easily be verified using induction after i).

Hence, using eq. (16) we have

(20) $$\mathrm{cov}_i(T_P, T_Q) = \sum_{\ell=0}^{n} \sigma_\ell^c \cdot s^{\ell+1} \cdot P_\ell(t) + A_i \sum_{j=0}^{i} \sum_{\ell=p}^{\infty} c_j/(\ell+k_j)\cdot s^{\ell+1} P_\ell(t).$$

The infinite sums in eq. (20) are equal to closed analytic expressions, F_j, (cf. Tscherning and Rapp (1974, section 8)):

(21) $$F_j = \sum_{\ell=p}^{\infty} 1/(\ell + j)\cdot s^{\ell+1} \cdot P_\ell(t).$$

Hence

(22) $$\mathrm{cov}_i(T_P, T_Q) = \sum_{\ell=0}^{n} \sigma_\ell^c \cdot s^{\ell+1} P_\ell(t) + A_i \sum_{j=0}^{i} c_j F_{k_j}.$$

The covariance function depends only on r and r′ through the quantity $s = (R_b^2/(r \cdot r'))$. So the effect of performing a differentiation with respect to r or r' will be a simple multiplication of the degree-variances with the degree plus an integer constant, and division with r or r'. We have for example

$$-D_r(\mathrm{cov}_i(T_P, T_Q)) = \mathrm{cov}_i(-D_r T, T_Q)$$

$$= \frac{1}{r}\left[\sum_{\ell=0}^{n} (\ell+1)\sigma_\ell^c \cdot s^{\ell+1} P_\ell(t) + A_i \sum_{j=0}^{i} c_j \sum_{\ell=p}^{\infty} (\ell+1)/(\ell+k_j)\cdot s^{\ell+1} P_\ell(t)\right]$$

$$= \frac{1}{r}\left[\sum_{\ell=0}^{n} (\ell+1)\sigma_\ell^c \cdot s^{\ell+1} P_\ell(t) + A_i \sum_{j=0}^{i} c_j(1-k_j) \sum_{\ell=p}^{\infty} 1/(\ell+k_j) s^{\ell+1} P_\ell(t)\right.$$

(23)
$$\left. + \sum_{j=0}^{i} c_j \sum_{\ell=p}^{\infty} s^{\ell+1} P_\ell(t))\right]$$

$$= \frac{1}{r}\left[\sum_{\ell=0}^{n} (\ell+1)\sigma_\ell^c s^{\ell+1} P_\ell(t) + A_i \sum_{j=0}^{i} c_j^1 F_{k_j}\right.$$

$$\left. + c_{i+1}^1 S_0),\right]$$

where we have put

$$c_j^1 = c_j(1-k_j) \text{ for } j \le i$$

$$c_{i+1}^1 = \sum_{j=0}^{i} c_j \quad \text{and}$$

$$S_0 = \sum_{\ell=p}^{\infty} s^{\ell+1} P_\ell(t)$$

Subsequent differentiations with respect to r and r' will produce terms such as

(24) $$c_{i+m+1}^q \cdot S_m, \quad m = 1, 2,$$

where

(25) $$S_m = \sum_{\ell=p}^{\infty} \ell^m \cdot s^{\ell+1} P_\ell(t)$$

and q is the total number of differentiations performed with respect to r and r'.

This is a fortunate situation, since the computation of radial derivatives of $\mathrm{cov}_i(T_P, T_Q)$ then only becomes slightly more complicated than the computation of the function itself.

Another fortunate fact is that some of the quantities c_i^q become zero. In the above equation **(23)**

$$c^1_{i+1} = \sum_{j=0} c_j$$

will be equal to zero (this is easily verified using eq. (19)). This means, that the term S_{q-1} never will occur, i.e. we will only have to compute the functions S_0, S_1, and S_2 when q = 4.

In several of the linear functionals we have chosen to consider (c.f. eq. (1)-(14)) the radial derivative will occur together with the evaluation functional applied to T and divided by r, T(P)/r. The only effect hereof is, that the degree-variances are multiplied by a constant $\ell+m$ instead of by .E+ 1. For the gravity anomaly we have for example m = -1.

For the covariance functions we will consider, up to four factors, (ℓ+ m) may occur. (Four factors will occur when the second order radial derivaiive occur in both P and Q). Table 1 shows the factors which occur for the functionals given by eq. (1)-(14).

Table 1

Table of factors (E+m) which occur, when applying the functionals given in eq. (1) (14).

Equation number	Factor	Number of differentiations
1	none	0
2	$\ell+1$	1
3	$\ell-1$	1
4	$(\ell-1)(\ell+2)$	2
5	(a+1)(ℓ+2)	2
6	none	0
7	none	0
8	$\ell-1$	1
9	a - 1	1
10	a+1	1
11	a+1	1
12	none	0
13	none	0
14	none	0

Let us therefore regard operators D, which multiply the degree-variances by ℓ+m and terms independent of E or k_j by zero, and let us then compute the factors c^q_j for $0<j \leq i+q$. These factors will occur when the operator is applied q times, each time with different m-values: , m,, m,, m_3 and m,.

We get:

(26) $$D_{m_1}\left(\frac{1}{\ell+k}\right) = \frac{\ell+m_1}{\ell+k} = 1 + \frac{m_1-k}{\ell+k} \;,$$

(27) $$D_{m_2}\left(D_{m_1}\left(\frac{(\ell+m_2)(m_1-k)}{\ell+k}\right)\right) = m_1 - k + \frac{(m_2-k)(m_1-k)}{\ell+k}$$

(28) $$D_{m_3}\left(D_{m_2}\left(D_{m_1}\left(\frac{1}{\ell+k}\right)\right)\right) = -k(\ell+m_3) + \frac{(\ell+m_3)(m_2-k)(m_1-k)}{\ell+k}$$

$$= -k\ell - km_3 + k^2 - m_2 k - m_1 k + m_1 m_2$$

$$+ \frac{(m_3-k)(m_2-k)(m_1-k)}{\ell+k}$$

$$= -k\ell + k(k-(m_1+m_2+m_3)) + \frac{(m_3-k)(k_2-k)(m_1-k)}{\ell+k} + m_1 m_2$$

and

(29) $$D_{m_4}\left(D_{m_3}\left(D_{m_2}\left(D_{m_1}\left(\frac{1}{\ell+k}\right)\right)\right)\right)$$

$$= (\ell+m_4)(-k\ell + k(k-(m_1+m_2+m_3))) + \frac{(\ell+m_4)(m_3-k)(m_2-k)(m_1-k)}{\ell+k}$$

$$= -k\ell^2 - \ell k((m_1+m_2+m_3+m_4)-k) - k(k(k-(m_1+m_2+m_3+m_4)) +$$

$$m_1 m_2 + m_1 m_3 + m_1 m_4 + m_2 m_3 + m_2 m_4 + m_3 m_4) + m_1 m_2 m_3 \;.$$

$$+ \frac{(m_4-k)(m_3-k)(m_2-k)(m_1-k)}{\ell+k}$$

In Table 2 we have then the coefficients c_j^q**:**

Table 2

Table of the coefficients c_j^q to the functions F_j and S_j for different order of differentiation **q**.

q	Function: F_j	S_0	S_1	S_2
	$j = k_0 - k_1$	$i+1$	$i+2$	$i+3$
0	c_j	0	0	0
1	$c_j(m_1 - k_j)$	0	0	0
2	$c_j^1(m_2 - k_j)$	$-\sum c_j k_j$	0	0
3	$c_j^2(m_3 - k_j)$	$\sum c_j k_j (k_j - \sum\limits_{n=1}^{q} m_n$	c_{i+1}^{q-1}	0
4	$c_j^3(m_4 - k_j)$	$\sum c_j(-k_j(k_j(k_j - \sum\limits_{n=1}^{4} m_n) + \sum\limits_{n=1}^{3} \sum\limits_{p=n+1}^{4} m_n m_p))$	c_{i+1}^{q-1}	c_{i+2}^{q-1}

The expressions for the coefficients c_{i+1}^q, c_{i+2}^q, c_{i+3}^q may be simplified somewhat, depending on the degree-variance model. For the different models we have the following equations, which may be verified by straight forward computations:

Model 1:

$$-\sum_{j=0}^{1} c_j k_j = 1,$$

$$\sum_{j=0}^{1} c_j k_j^2 = -\sum_{j=0}^{1} k_j,$$

$$-\sum_{j=0}^{1} c_j k_j^3 = k_0^2 + k_0 k_1 + k_1^2 ,$$

Model 2:

$$-\sum_{j=0}^{a} c_j k_j = 0,$$

$$\sum_{j=0} c_j k_j^2 = 1 \,,$$

$$-\sum_{j=0}^{2} c_j k_j^3 = -\sum_{j=0}^{2} k_j \,,$$

Model 3:

$$-\sum_{j=0}^{3} c_j k_j = \sum_{j=0}^{3} c_j k_j^2 = 0$$

$$-\sum_{j=0}^{3} c_j k_j^3 = 1$$

Putting

$$I = \sum_{n=1}^{4} m_n \quad \text{and} \quad J = \sum_{n=1}^{3} \sum_{p=n+1}^{4} m_n m_p$$

we have

Model 1:

$$c_2^2 = c_3^3 = c_4^4 = 1,$$

$$c_2^3 = c_3^4 = -k_0 k_1 + I,$$

$$c_2^4 = k_0^2 + k_0 k_1 + k_1^2 - (k_0 + k_1)\, I + J,$$

Model 2:

$$c_3^2 = c_4^3 = c_5^4 = 0,$$

$$c_3^3 = c_4^4 = 1,$$

$$c_3^4 = -\sum_{j=0}^{2} k_j + I,$$

Model 3:

$$c_4^2 = c_5^2 = c_6^2 = c,^2 = c_5^4 = 0$$

$$c_4^4 = 1$$

Let us use the equations and the Table to derive the covariance function of the gravity anomalies for Model 1 having $p = 3$, $k = -2$, $k = -1$, for Model 2 having in addition $k_2 = i$ and for Model 3 having in addition $k_3 = j$. Using eq. (3) we have

$$\mathrm{cov}_1(\Delta g, \Delta g') = (-D_r - \frac{2}{r} EV_P) \circ (-D_r' - \frac{2}{r'} EV_Q)\, \mathrm{cov}_1(T_P, T_Q),$$

where EV_P, EV_Q are the evaluation functionals at P, Q respectively, i.e. $EV_P(T) = T(P)$ and $EV_Q(T) = T(Q)$.

For Model 1 we have $c_0 = 1$, $c_1 = -1$, $m, = m_2 = -1$ and hence

$$c_0^2 = +1(-1+2)(-1+2) = +1,$$

$$c_1^2 = -1(-1+1)(-1+1) = 0,$$

$$c_2^2 = +(-2 \cdot (-1) - (-1)) = 1,$$

$$\mathrm{cov}_1(Ag, Ag') = A_1(F-, + S_0) \cdot \frac{1}{r \cdot r'} .$$

For Model 2 we have $c_0 = 1/(i+2)$, $c, = -1/(i+1)$, $c, = 1/((i+1)(i+2))$ and again $m_1 = m_2 = -1$: Hence

$$c_0^2 = c_1,\ c_1^2 = 0,\ c_2^2 = (i+1)/(i+2)$$

$$c_3^2 = 2/(i+2) - 1/(i+1) - i/((i+1)(i+2)) = 0$$

and

$$\mathrm{cov}_2(\Delta g, \Delta g') = A_2 \left(\frac{1}{i+2} F_{-2} + \frac{i+1}{i+2} F_1 \right) \frac{1}{r \cdot r'},$$

cf. Tscherning and Rapp (1974, eq. (132)).

Finally for Model 3 we have

$$c_0 = 1/((i+2)(j+2)),\quad c_1 = -1/((i+1)(j+1)),$$

$$c_2 = 1/((i+1)(i+2)(j-i)),\quad c_3 = -1/((j+1)(j+2)(j-i)),$$

hence

$$c_0^a = c_0,$$

$$c_1^2 = 0$$

$$c_2^a = 1/((i+2)(j-i))$$

$$c_3^o = 1/((j+2)(j-1))$$

$$c_4^a = 2/((i+2)(j+2)-1/((i+1)(j+1))$$

$$-i/((i+1)(i+2)(j-1))-j/((j+1)(j+2)(j-1))$$

$$= 0$$

and

$$\mathrm{cov}_3(Ag,\ Ag') = A_3(F_{-1}/((i+2)(j+2))-F_{-1}/((i+1)(j+1))$$
$$+F_i/((i+2)(j-i))+F_j/((j+2)(j-i)))\frac{1}{r \cdot r'},$$

cf. Tscherning and Rapp (1974, eq. (141)).

We will end this section by writing down the analytic expressions for F_j and S_j. The basic equation is

(30) $$S_0 = \sum_{\ell=p}^{\infty} s^{\ell+1} P_\ell(t) = s/L - \sum_{\ell=0}^{p-1} s^{\ell+1} P_\ell(t),$$

where

(31) $$L = (1-2st+s^2)^{\frac{1}{2}}.$$

Introducing the quantities

(32) $$N = 1+L-st$$

$$M = 1-L-st$$

and computing the derivatives

$$D_s(1/L) = \sum_{\ell=1}^{\infty} \ell s^{\ell-1} P_\ell(t) = -D_s L/L^2 = (t-s)/L^3,$$

$$D_s\,(s(t-s)/L^3) = \sum_{\ell=1} \ell^{2} P_{\ell}(t)\, s^{\ell-1}$$

$$= ((t-2s)L^3 - s(t-s)(s-t)L \cdot 3)/L^6$$

$$= (t-2s)/L^3 + (s-t)^2 s \cdot 3/L^5$$

$$= (t-2s+3s)/L^3 + 3s\,(t^2-1)/L^5$$

$$= (t+s)/L^3 + 3s(t^2-1)/L^5\,,$$

we get

(34) $$S_1 = \sum_{\ell=p}^{\infty} \ell\, s^{\ell+1} P_{\ell}(t) = s^{2}(t-s)/L^3 - \sum_{\ell=1}^{p-1} \ell\, s^{\ell+1} P_{\ell}(t)$$

and

(35) $$S_2 = \sum_{\ell=p}^{\infty} \ell^2 P_{\ell}(t) s^{\ell+1} = s^2((t+s)/L^3 + 3s(t^2-1)/L^5) - \sum_{\ell=1}^{p-1} \ell^2 P_{\ell}(t) s^{\ell+1}.$$

Using Tscherning and Rapp (1974, eq. (99), (100) and (96)) we get the following equations

(36) $$F_1 = \ln\,(1 + (1 - s L)) - \sum_{\ell=0}^{p-1} 1/(\ell+1) s^{\ell+1} P_{\ell}(t)$$

(37) $$F_2 = (L - 1 + t \cdot F_1^0)/s - \sum_{\ell=0}^{p-1} 1/(\ell+2) s^{\ell+1} P_{\ell}(t)$$

and the recursion formulae

(38) $$F_{i+1}^0 = (L + (2i-1)\, t \cdot F_i^0 - (i-1)/s \cdot F_{i-1}^0)/(s-i),$$

where the superscript 0 indicates, that $p = 0$ in the equation, i.e.

(39) $$F_{i+1} = F_{i+1}^0 - \sum_{\ell=0}^{p-1} 1/(\ell+i+1) \cdot s^{\ell+1} P_{\ell}(t).$$

From Ibid. (eq. (84)- (87)) we get

(40) $$F_0 = s \cdot \ln(2/N) - \sum_{\ell=1}^{p-1} 1/\ell \cdot s^{\ell+1} P_\ell(t),$$

(41) $$F_{-1} = s(M + ts \cdot \ln(2/N)) - \sum_{\ell=2}^{p-1} 1 \quad 1 \quad {}^{\ell+1} P(t),$$

(42) $$F_{-2} = s(M(3ts+1)/2 + s(P_2(t)s \cdot \ln(2/N) + s(1-t^2)/4))$$

$$- \sum_{\ell=3}^{p-1} 1/(\ell-2)s^{\ell+1} P_\ell(t).$$

(There does exist a recursion formula similar to equation (38) for negative j, but we will not write it down, because we will not consider models, where j is less than –2.)

Combining all these equations, we are now able to compute radial derivatives of up to second order of the covariance functions corresponding to model 1, 2, and 3.

3. Derivatives with respect to the latitude and the longitude.

We will now consider the computation of the covariance between quantities where at least one includes differentiation with respect to one of the coordinates φ, X, Q or X'. Let us suppose, that the necessary differentiations with respect to r or r' have been executed. Let us for example consider the computation of

$$\mathrm{cov}(D_\varphi D_X T/(r^2 \cos p), Ag') = 1/(r^2 \cos\varphi) D_\varphi D_\lambda \mathrm{cov}(T_P, \Delta g').$$

The quantity cov $(T_P, \Delta g')$ will depend on the coordinates φ, X, φ', X' through the variable

(43) $$t = \cos\psi = \sin\varphi \sin\varphi' + \cos\varphi \cos\varphi' \cos(\Delta\lambda),$$

$$\Delta\lambda = \lambda' - X.$$

Therefore, the computation of the derivatives with respect to the latitude or longitude, can be divided in a computation of the derivatives of t (up to order four) of a radial derivative of $\mathrm{cov}(T_P, T_Q)$ and a subsequent multiplication with derivatives of t with respect to the latitude and the longitude. We will in the following denote the quantity which remains to be differentiated with respect to t by K. An integer superscript or a number of apostrophes will indicate the order of differentiation, i,e. $D_t^2 K = K'' = K$,.

It is worthwhile to systematize the derivatives of t in some way. We will do this by associating different integers with the different kinds of differentiations. Basically we will associate the integers 0 and 1 with no differentiation at all, and the integers 2 and 3 with differentiation with respect to the latitude (in P or Q) and the longitude (in P or Q), respectively. The idea is now, to associate with the differentiations with respect to e. g. φ and λ the sum of the two integers, namely 5, with this second order derivative.

The derivatives may be computed in both P and Q, but the order of differentiation is maximally two, and the total number of differentiations is hence maximally four. The kinds of differentiation may then be characterized by four integers:

(i) kind of first differentiation in P (none, D_φ, D_λ)

(j) " second " in P (none and no first, none, D_φ, D_λ)

(k) " first " in Q (none, D_φ, D_λ)

(m) second " in Q (none and no first, none, D_φ, D_λ).

The following algorithm will then associate a unique integer d with the appropriate derivative of t:

(a) only one first order derivative:

in P: $d = i$
in Q: $d = 6 \cdot (k-1) + 1$

(b) two derivatives:

both in P: $d = i + j$
both in Q: $d = 6 \cdot (k + m - 1) + 1$
in P and Q: $d = i + 6\,(k-1)$

(c) three derivatives:

onlyone in P: $d = i + 6\,(k + m - 1)$
only one in Q: $d = i + j + 6(k - 1)$

and

(d) four derivatives:

$d = i + j + 6\,(k + m - 1).$

The integer 17, will for example, be associated with the derivative $D_{\varphi} D_{\lambda} D_{\lambda'}(t)$. The relationship between the derivatives and the integer d is shown in Table 3.

Table 3

The integer d associated with the derivatives of t with respect to φ, λ, φ' and λ':

	none in Q	$D_{\varphi'}$	$D_{\lambda'}$	$D^2_{\varphi'}$	$D_{\varphi'} D_{\lambda'}$	$D^2_{\lambda'}$
none in P	1	7	13	19	25	31
D_{φ}	2	8	14	20	26	32
D_{λ}	3	9	15	21	27	33
D^2_{φ}	4	10	16	22	28	34
$D_{\varphi} D_{\lambda}$	5	11	17	23	29	35
D^2_{λ}	6	12	18	24	30	36

We will now write down the general equations for the derivatives of K with respect to φ, λ, φ' and λ'. Let the variables x, y, z and v denote any one of these (though only two of them can be the same variable in our case).

(43) $$D_x K = D_x t \cdot K_1 ,$$

(44) $$D_x D_y K = D_x D_y t \cdot K_1 + D_x t \cdot D_y t \cdot K_2 ,$$

(45) $$D_x D_y D_z K = D_{xyz} t \cdot K_1 + D_{xy} t \cdot D_z t \cdot K_2 + (D_{xz} t \cdot D_y t + D_x t \cdot D_{yz} t) K_2 + D_x t \cdot D_y t \cdot D_z t K_3 ,$$

(46) $$D_x D_y D_z D_v K = D_{xyzv} t \cdot K_1 + [D_{xyv} t \cdot D_z t + D_{xzv} t \cdot D_y t + D_{yzv} t \cdot D_x t + D_{xyz} t \cdot D_v t + D_{xy} t \cdot D_{zv} t + D_{xz} t \cdot D_{yv} t + D_{yz} t \cdot D_{xv} t] \cdot K_2 + [D_{xy} t \cdot D_z t \cdot D_v t + D_{xz} t \cdot D_y t \cdot D_v t + D_{yz} t \cdot D_x t \cdot D_v t + D_{xv} t \cdot D_y t \cdot D_z t + D_{yv} t \cdot D_x t \cdot D_z t + D_{zv} t \cdot D_x t \cdot D_y t] \cdot K_3 + D_x t \cdot D_y t \cdot D_z t \cdot D_v t \cdot K_4$$

We note, that in order to compute a derivative of e.g. 3'th order, all the de ivatives of 3'th and lower order of t are needed Therefore, when we e. g. differentiate K with respect to (φ, a', X'), the derivatives of t with respect to all three variables, (φ, X'), (ω', X'), (a, ω'), ω, ω' and χ' are needed. This enables us to write down a general algorithm based on the assignment of the integers 0, 1, 2 and 3 to the variables i, j, k and m. But we have to introduce two more variables j, and m, as to distinguish between the kind of differentiation we are performing in itself (φ, ω', X' above) and the lower order differentiations.

Let us suppose that the derivatives of t are evaluated and stored in an array d with subscripts from 1 to 36, cf. Table 3. The integers i, j, k and m are then associated with the variables x, y, z and v used in eq. (43)-(46).

We then have:

(a) Only one differentiation (either i or k are equal to 2 or 3, respectively 1 and j, m are zero):

$$D_x K = d(i+6(k-1)) \cdot K_1 ,$$

(b) two differentiations (j or m are equal to zero, in which case j1 or m1 will be one and otherwise equal to j, m respectively).

$$D_{xy} K = d(i)\ d(j1)d(6(k-1)+1)d(6(m1-1)+1)K_2 + d(i+j+6(k+m-1))K_1 ,$$

(c) three differentiations (either j or m are equal to zero, in which case j1 or m1 will be one),

$$D_{xyz} K = d(i+j+6(k+m-1))K_1 + (d(i+j)d(6(k+m-1)+1) + d(i+6(k-1))d(j1+6(m1-1))$$
$$+ d(i+6(m1-1))d(j1+6(k-1)))K_2 + d(i)d(j1)d(6(k-1)+1) \cdot d(6(m1-1)+1)K_3$$

(d) four differentiations (i; j, k and m are all greater than 1),

$$D_{xyzv} K = d(i+j+6(k+m-1))K_1 + (d(i+j+6(k-1))d(6(m-1)+1)+d(i+6(k+m-1)) \cdot d(j)$$
$$+ d(j+6(k+m-1))d(i)+d(i+j+6(m-1))d((k-1)6+1)$$
$$+ d(i+j)d(6(k+m-1)+1)+d(i+6(k-1))d(j+6(m-1))$$
$$+ d(i+6(m-1))d(j+6(k-1)))K_2$$
$$+ (d(i+j)d(6(k-1)+1)d(6(m-1)+1)+d(i+6\ (k-1))d(j)d(6(m-1)+1)$$
$$+ d(i+6(m-1))d(j)d(6(k-1)+1)+d(j+6(k-1))d(i)d(6(m-1)+1)$$

$$+ \cdot d(j+6(m-1))d(i)d(6(k-1)+1))d(6(k+m-1)+1)d(i)d(j))K_3$$

$$+ d(i)d(j)d(6(k-1)+1)d(6(m-1)+1) \cdot K_4 .$$

We will now explicitly calculate the 36 different de ivatives of t, i. e. the values of the array d.

$d(1) = 1$

$d(2) = D_\varphi t = \cos\varphi \sin\varphi' - \sin\varphi \cos\varphi' \cos(\Delta\lambda) = cs - scc,$

$d(3) = D_\lambda t = \cos\varphi \cos\varphi' \sin(\Delta\lambda) = ccs,$

$d(4) = D^2_\varphi t = -\sin\varphi \sin\varphi' - \cos\varphi \cos p' \cos(AA) = -t,$

$d(5) = D_\varphi D_\lambda t = -\sin\varphi \cos\varphi' \sin(\Delta\lambda) = -scs,$ r

$d(6) = D^2_A t = -\cos\varphi \cos\varphi' \cos(AA) = -ccc,$

$d(7) = D_{\varphi'} t = \sin\varphi \cos\varphi' - \cos\varphi \sin\varphi' \cos(AA) = sc-csc,$

$d(8) = D_\varphi D_{\varphi'} t = \cos\varphi \cos\varphi' + \sin\varphi \sin\varphi' \cos(\Delta\lambda) = cc+ssc,$

$d(9) = D_\lambda D_{\varphi'} t = -\cos\varphi \sin\varphi' \sin(AA) = -css,$

$d(10) = D^2_\varphi D_{\varphi'} t = -\sin\varphi \cos\varphi' + \cos\varphi \sin\varphi' \cos(\Delta\lambda) = -sc+csc,$

$d(11) = D_\lambda D_\varphi D_{\varphi'} t = \sin\varphi \sin\varphi' \sin(\Delta\lambda) = sss,$

$d(12) = D^2_\lambda D_{\varphi'} t = \cos\varphi \sin\varphi' \cos(AA) = csc,$

$d(13) = D_{A'} t = -\cos\varphi \cos\varphi' \sin(AA) = -ccs,$

$d(14) = D_\varphi D_{\lambda'} t = \sin\varphi \cos\varphi' \sin(\Delta\lambda) = scs,$

$d(15) = D_\lambda D_{\lambda'} t = \cos\varphi \cos\varphi' \cos(AA) = ccc,$

$d(16) = D^2_\varphi D_{\lambda'} t = \cos\varphi \cos\varphi' \sin(\Delta\lambda) = ccs,$

$d(17) = D_\varphi D_\lambda D_{\lambda'} t = -\sin\varphi \cos\varphi' \cos(\Delta\lambda) = -scc,$

$d(18) = D^2_\lambda D_{\lambda'} t = \cos(p\cos\varphi' \sin(AA) = ccs,$

$d(19) = D^2_{\varphi'} t = -\sin\varphi \sin\varphi' - \cos\varphi \cos\varphi' \cos(AA) = -t,$

$$d(20) = D_{\varphi} D^{2}_{\varphi'} - - \cos\varphi \sin\varphi' + \sin\varphi \cos\varphi' \cos(\Delta\lambda) = -cs + scc$$

$$d(21) = D_{\lambda} D^{*}_{\varphi'} t = -\cos\varphi \cos\varphi' \sin(AA) = -ccs,$$

$$d(22) = D^{2}_{\varphi} D^{2}_{\varphi'} t = \sin\varphi \sin\varphi' + \cos\varphi \cos\varphi' \cos(AA) = t,$$

$$d(23) = D''_{\lambda} D_{\varphi} D^{2}_{\varphi'} t = +\sin\varphi \cos\varphi' \sin(AA) = scs,$$

$$d(24) = D^{2}_{X} D^{*}_{\varphi'} t = +\cos\varphi \cos\varphi' \cos(AA) = ccc,$$

$$d(25) = D_{\varphi'} D_{\lambda'} t = \cos\varphi \sin\varphi' \sin(\Delta\lambda) = css,$$

$$d(26) = D_{\varphi} D_{\varphi'} D_{\lambda'} t = -\sin\varphi \sin\varphi' \sin(\Delta\lambda) - sss,$$

$$d(27) = D_{\lambda} D_{\varphi'} D_{\lambda'} t = -\cos\varphi \sin\varphi' \cos(AA) = -csc,$$

$$d(28) = D^{*}_{\varphi} D_{\varphi'} D_{\lambda'} t = -\cos\varphi \sin\varphi' \sin(\Delta\lambda) = -css,$$

$$d(29) = D_{\varphi} D_{\lambda} D_{\varphi'} D_{\lambda'} t = \sin\varphi \sin\varphi' \cos(AA) = ssc,$$

$$d(30) = D^{2}_{\lambda} D_{\varphi'} D_{\lambda'} t = -\cos\varphi \sin\varphi' \sin(\Delta\lambda) = -css,$$

$$d(31) = D^{2}_{\lambda'} t = -\cos\varphi \cos\varphi' \cos(\Delta\lambda) = -ccc,$$

$$d(32) = D_{\varphi} D_{\lambda'} t = \sin\varphi \cos\varphi' \cos(AA) = scc,$$

$$d(33) = D_{\lambda} D^{2}_{\lambda'} t = -\cos\varphi \cos\varphi' \sin(AA) = -ccs,$$

$$d(34) = D^{2}_{\varphi} D^{2}_{\lambda'} t = \cos\varphi \cos\varphi' \cos(AA) = ccc,$$

$$d(35) = D_{\lambda} D_{\varphi} D^{2}_{\lambda'} t = \sin\varphi \cos\varphi' \sin(\Delta\lambda) = scs,$$

$$d(36) = D^{2}_{X} D^{2}_{X'} t = \cos\varphi \cos\varphi' \cos(\Delta\lambda) = ccc,$$

where $cs = \cos\varphi \sin\varphi'$, $cc = \cos\varphi \cos\varphi'$ etc. In Table 4 the derivatives are presented on a form corresponding to Table 3.

Table 4

Derivatives of t with respect to φ, X, φ' and X'

d(i+j+6(k+m))	k+m					
i+j	1	2	3	4	5	6
1	1	sc -csc	-ccs	-t	css	-ccc
2	cs -scc	cc ssc	scs	-cs scc	-sss	scc
3	ccs	-css	ccc	-ccs	-csc	-ccs
4	-t	-sc csc	ccs	t	-css	ccc
5	-scs	sss	-scc	scs	ssc	scs
6	-ccc	csc	ccs	ccc	-css	ccc

The general form of K is

$$K = \Big(\sum_{j=0}^{1} c_j^q F_{k_j} + c_{i+1}^q S_0 + c_{i+2}^q S_1 + c_{i+3}^q S_2 \Big) / \left(r^k \cdot (r')^m \right) \tag{47}$$

+ a sum of a finite Legendre series,

cf. section 2, where k and m are the order of differentiation with respect to r and r' and $q = k + m$. The term S_2 is only present when $k = m = 2$ and will hence never have to be differentiated with respect to t. In a similar way we see, that only when q is equal to 3, is it necessary to compute $D_t S_1$ and only when q is equal to 2 it may be necessary to compute $D_t^2 S_0$.

Table 5 shows, which terms we will need to differentiate in the different degree-variance models and what is the maximal order of differentiation we need to compute.

Table 5

Order of differentiation and kind of quantities which have to be differentiated with respect to t

Maximal order of differentiation with respect to t	Order of radial differentiation, q				
	0	1	2	3	4
	4	3	2	1	0
Degree-variance model					
1	F_j	F_j	F_j, S_0	F_j, S_0, S_1	F_j, S_0, S_1, S_2
2	F_j	F_j	F_j	F_j, S_0	F_j, S_0, S_1
3	F_j	F_j	F_j	F_j	F_j, S_0

We will hence have to compute expressions for the 1'th to 4'th derivatives of F_j, of the l'th and second derivatives of S_0 and of the 1'th derivative of S, and not to forget, the 1'st to 4'th derivatives of the sum of the finite Legendre series.

The derivatives of the sum of a finite Legendre series

(48) $$S = \sum_{\ell=0}^{n} a_\ell \cdot s^{\ell+1} P_\ell(t)$$

may be computed easily using a simple recursion algorithm, cf. Tscherning and Rapp (1974, p. 67). For

$$e_\ell = (2\ell+1)/(\ell+1) \cdot s, \qquad f_{\ell+1} = -(\ell+1)/(\ell+2) \cdot s^2$$

and

(49) $$b_\ell = e_\ell t\, b_{\ell+1} + f_{\ell+1} b_{\ell+2} + a_\ell \text{ with } b_{n+1} = b_{n+2} = 0,$$

we have

(50) $$S = b_0 \cdot s.$$

The derivatives of S with respect to t are then computed by a recursion algorithm obtained by differentiating eq. (49):

$$b_\ell^1 = e_\ell(b_{\ell+1}^1 + tb_{\ell+1}) + f_{\ell+1} \cdot b_{\ell+2}^1$$

$$b_\ell^2 = e_\ell(2b_{\ell+1}^1 + tb_{\ell+1}^2) + f_{\ell+1} \cdot b_{\ell+2}^2$$

and generally with the superscript q indicating the order of differentiation to t:

$$(51) \qquad b_\ell^q = e_\ell(q \cdot b_{\ell+1}^{q-1} + t \cdot b_{\ell+1}^q) + f_{\ell+1} \cdot b_{\ell+2}^q \; ,$$

and still

$$(52) \qquad S^q = b_0^q \cdot s.$$

For the expressions F_j and S_j we must do the hard work of differentiating the expressions up to four times.

Let us first differentiate $1/L$:

$$D_t(1/L) = s/L^3,$$

$$D_t^2(1/L) = 3s^2/L^5.$$

Hence from eq. (30) we have

$$(53) \qquad D_t S_0 = s^2/L^3 - \sum_{\ell=0}^{p-1} s^{\ell+1} P'_\ell(t)$$

$$(54) \qquad D_t^2 S_0 = 3s^3/L^5 - \sum_{\ell=0}^{p-1} s^{\ell+1} P''_\ell(t)$$

and from eq. (34)

$$(55) \qquad D_t S_1 = s^2(1/L^3 + 3(t-s)s/L^5) - \sum_{\ell=1}^{p-1} \ell s^{\ell+1} P_\ell(t).$$

From the computation of the derivatives of F_j we will need the following derivatives.

$$(56) \qquad D_t(1/(L^i N^j)) = s(i/L^{i+2} N^j) + j/(L^{i+1} N^{j+1}) + j/(L^i N^{j+1})),$$

where we have used

$$D_t L = -s/L \text{ and } D_t N = -s/L - s.$$

From Tscherning and Rapp (1974, section 8) we have the following derivatives of F_j (where we have left out the finite series part);

(57) $$D_t F_0 = s \cdot D_t (\ln(2/N)) = s^2(1/(LN) + 1/N),$$

(58) $$D_t^2 F_0 = s^3((N+L)/(L^3N^2) + (2+L)/(LN^2))$$
$$= s^3(1/L^3N) + 1/(L^2N^2) + 2/(L \cdot N^2) + 1/N^2).$$

Using eq. (56) we get

(59) $$D_t^3 F_0 = s^4[3/(L^5N) + 1/(L^3N^2) + 1/(L^4N^2) + 2/(L^4N^2)$$
$$+ 2/(L^2N^3) + 2/(L^3N^3) + 2/(L^3N^2) + 4/(L \cdot N^3)$$
$$+ 4/(L^2N^3) + 2/N^3 + 2/(LN^3)]$$
$$= s^4[3/(L^5N) + 3/(L^3N^2) + 3/(L^4N^2) + 6/(L^2N^3)$$
$$+ 2/(L^3N^3) + 6/(LN^3) + 2/N^3]$$
$$= s^4[(3/L^5 + (3(1+1/L)/L^3 + 2(1+(3+(3+1/L)/L)/L)/N)/N)/N)]$$

(60) $$D_t^4 F_0 = 3s^5[5/(L^7N) + 1/(L^5N^2) + 1/(L^6N^2) + 3/(L^5N^2)$$
$$+ 2/(L^3N^3) + 2/(L^4N^3) + 4/(L^6N^2) + 2/(L^4N^3) + 2/(L^5N^3)$$
$$+ 4/(L^4N^3) + 6/(L^2N^4) + 6/(L^3N^4) + 2/(L^5N^3) + 2/(L^3N^4)$$
$$+ 2/(L^4N^4) + 2/(L^3N^3) + 6/(LN^4) + 6/(L^2N^4) + 2/N^4 + 2/(LN^4)]$$
$$= 3s^5[5/L^7N) + 4/(L^5N^2) + 5/(L^6N^2) + 4/(L^3N^3)$$
$$+ 8/(L^4N^3) + 4/(L^5N^3) + 12/(L^2N^4) + 8/(L^3N^4)$$
$$+ 2/(L^4N^4) + 8/(LN^4) + 2/N^4]$$
$$= 3s^5[(5/L^7 + ((4+5/L)/L^5 + ((4+(8+4/L)/L)/L^3$$
$$+ (2+(8+(12+(8+2/L)/L)/L)/L)/N)/N)/N)/N].$$

We then have using eq. (41) and (42) (and eq. (40) without the finite sum):

(61) $$D_t F_{-1} = s(D_t M + t \cdot D_t F_0 + F_0) - \sum_{\ell=2}^{p-1} 1/(\ell-1)s^{\ell+1} D_t P_\ell(t),$$

(62) $$D_t^2 F_{-1} = s(D_t^2 M + t D_t^2 F_0 + 2D_t F_0) - \sum_{\ell=2}^{p-1} 1/(\ell-1)s^{\ell+1} D_t^2 P_\ell(t),$$

(63) $$D_t^3 F_{-1} = s(D_t^3 M + tD_t^3 F_0 + 3D_t^2 F_0) - \sum_{\ell=2}^{p-1} 1/(\ell-1)s^{\ell+1} D_t^3 P_\ell(t),$$

(64) $$D_t^4 F_{-1} = s(D_t^4 M + tD_t^4 F_0 + 4D_t^3 F_0) - \sum_{\ell=2}^{p-1} 1/(\ell-1)s^{\ell+1} D_t^4 P_\ell(t),$$

(65) $$D_t F_{-2} = s[D_t M(3ts+1)/2 + M3s/2 + s(P_2(t)D_t F_0 + 3tF_0$$
$$-st/2)] - \sum_{\ell=3}^{p-1} 1/(\ell-2)s^{\ell+1} D_t P_\ell(t),$$

(66) $$D_t^2 F_{-2} = s[D_t^2 M(3ts+1)/2 + 3sD_t M + s(P_2(t)D_t^2 F_0 + 6tD_t F_0$$
$$+3F_0 - s/2)] - \sum_{\ell=3}^{p-1} 1/(\ell-2)s^{\ell+1} D_t^2 P_\ell(t),$$

(67) $$D_t^3 F_{-2} = s[D_t^3 M(3ts+1)/2 + 9sD_t^2 M/2 + s(P_2(t)D_t^3 F_0$$
$$+9(tD_t^2 F_0 + D_t F_0))] - \sum_{\ell=3}^{p-1} 1/(\ell-2)s^{\ell+1} D_t^3 P_\ell(t),$$

(68) $$D_t^4 F_{-2} = s[D_t^4 M(3ts+1)/2 + 6sD_t^3 M + s(P_2(t)D_t^4 F_0 + 12tD_t^3 F_0$$
$$+18D_t^2 F_0)] - \sum_{\ell=4}^{p-1} 1/(\ell-2)s^{\ell+1} D_t^4 P_\ell(t),$$

where

(69a) $D_t M = -s + s/L,$

(69b) $D_t^2 M = s^2/L^3,$

(69c) $D_t^3 M = 3s^3/L^5,$

(69d) $D_t^4 M = 15s^4/L^7.$

From Tscherning and Rapp **(1974,** eq. **(101))** and from eq. **(36)** we **have**

$$D_t F_1 = s^2/(L \cdot N) - \sum_{\ell=1}^{p-1} 1/(\ell+1)s^{\ell+1} D_t P_\ell(t),$$

$$D_t^2 F_1 = s^3(1/(L^2N^2) + 1/(LN^2) + 1/(NL^3))$$

$$- \sum_{\ell=2}^{p-1} 1/(\ell+1)s^{\ell+1} D_t^2 P_\ell(t)$$

and hence, using eq. (56), **we have**

(70) $$D_t^3 F_1 = s^4[2/(L^4N^2) + 2/(L^2N^3) + 2/(L^3N^3)$$

$$+1/(L^3N^2) + 2/(LN^3) + 2/(L^2N^3) + 1/(N^2L^3) + 1/(N^2L^4) + 3/(NL^5)]$$

$$- \sum_{\ell=3}^{p-1} 1/(\ell+1)s^{\ell+1} D_t^3 P_\ell(t)$$

$$= s^4[3/(L^4N^2) + 4/(L^2N^3) + 2/(L^3N^3) + 2/(L^3N^2)$$

$$+2/(LN^3) + 3/(NL^5)] - \sum_{\ell=3}^{p-1} 1/(\ell+1)s^{\ell+1} D_t^3 P_\ell(t)$$

$$= s^4[(3/L^4 + ((2+3/L)/L^2 + (2+(4+2/L)/L)/N)/N)/(NL)]$$

$$- \sum_{\ell=3}^{p-1} 1/(\ell+1)s^{\ell+1} D_t^3 P_\ell(t),$$

$$(71)\qquad D_t^4 F_1 = s^5[12/(L^6N^2)+6/(L^4N^3)+6/(L^5N^3)$$
$$+8/(L^4N^3)+12/(L^2N^4)+12/(L^3N^4)+6/(L^5N^3)+6/(L^3N^4)$$
$$+6/(L^4N^4)+6/(L^5N^3)+4/(L^3N^3)+4/(L^4N^3)+2/(L^3N^3)$$
$$+6/(LN^4)+6/(L^2N^4)+15/(L^7N)+3/(L^5N^2)+3/(L^6N^2)]$$
$$-\sum_{\ell=4}^{p-1} 1/(\ell+1)s^{\ell+1} D_t^4 P_\ell(t)$$

$$= 3s^5[5/(L^7N)+5/(L^6N^2)+6/(L^4N^3)+4/(L^5N^3)+6/(L^2N^4)$$
$$+6/(L^3N^4)+2/(L^4N^4)+2/(L^3N^3)+2/(LN^4)+3/(L^5N^2)]$$
$$-\sum_{\ell=4}^{p-1} 1/(\ell+1)s^{\ell+1} D_t^4 P_\ell(t)$$

$$=3s^5[(5/L^6+((3+5/L)/L^4+((2+(6+4/L)/L)/L^2$$
L)/N
(2 $+(6+(6+2/L)/L)/(NL))/N)/N)/(NL)]$
$$-\sum_{\ell=4}^{p-1} 1/(\ell+1)s^{\ell+1} D_t^4 P_\ell(t).$$

Then we have from eq. (37), (where the superscript 0 again denotes that $p=0$ in the equation for F_1^0)

$$(72)\qquad D_t F_2 = (-s/L+tD_t F_1^0+F_1^0)/s-\sum_{\ell=1}^{p-1} 1/(\ell+2)s^{\ell+1} D_t P_\ell(t),$$

$$(73)\qquad D_t^2 F_2 = (-s^2/L^3+tD_t^2 F_1^0+2D_t F_1^0)/s-\sum_{\ell=2}^{p-1} 1/(\ell+2)s^{\ell+1} D_t^2 P_\ell(t).$$

$$(74)\qquad D_t^3 F_2 = (-3s^3/L^5+tD_t^3 F_1^0+3D_t^2 F_1^0)/s-\sum_{\ell=3}^{p-1} 1/(\ell+2)s^{\ell+1} D_t^3 P_\ell(t),$$

$$(75)\qquad D_t^4 F_2 = (-15s^4/L^7 + t \cdot D_t^4 F_1^0 + 4D_t^3 F_1^0)/s - \sum_{\ell=4}^{p-1} 1/(\ell+2)s^{\ell+1} D_t^4 P_\ell(t),$$

and finally the recursion algorithm (cf. eq. (39)):

$$(76)\qquad D_t^k F_{i+1}^0 = (D_t^k L + (2i-1)(kD_t^{k-1} F_i^0 + tD_t^k F_i^0)$$

$$-(i-1)/s \cdot D_t^k F_{i-1}^0)/(i \cdot s),$$

where

$$(77a)\qquad D_t L = -s/L$$

$$(77b)\qquad D_t^2 L = -s^2/L^3$$

$$(77c)\qquad D_t^3 L = -3s^3/L^5$$

$$(77d)\qquad D_t^4 L = -15s^4/L^7.$$

Combining all the equations given in section 2 and 3, we are now able to compute the covariance functions of the quantities (1)- (14) corresponding to the degree-variance models 1, 2 and 3. The practical set up of the computations is described in the following section.

4. The subroutine COVAX.

For the computation of the covariances a FORTRAN IV subroutine named COVAX has been designed and tested on the IBM system/370 computer of the Ohio State University Instruction and Research Computer Center. Tests were carried out for all three degree-variances models, for all combinations of the quantities given by *eq*, (1)- (14) and for a representative sample of points P and Q outside the Bjeshammar-sphere.

Fortunately a very good numerical control is available due to the fact, that the covariance functions where one of the quantities is the anomalous potential are harmonic functions. A numerical evaluation of the Laplace equation using eq, (15) will therefore give a result, which will indicate the order of the round off errors, We will in this way have a check of the numerical evaluation of all covariances between quantities given by eq. (1)- (14) in one point and by eq. (2), (5), (6), (12) and (14) in the other. For the other covariance functions only errors occurring while using these e. g. in least squares collocation may unveil programming errors.

The tests showed, that the round off errors depended on the complexity of the used degree-variance model. But only in extreme cases did the relative error exceed 10^{-6}. This occurred when big values of $k_j (>500)$ were used in the model degree-variances (*eq.* (17)) and when the difference between the radius of the Bjerhammar-sphere and the mean radius of the Earth was small (500 m). The round off errors did generally decrease for increasing altitude. However, when the subroutine was tested with one of the quantities k_j = 1500 and with the points of evaluation both situated in a height of 250 km, overflow occurred.

This was caused by applying the recursion formulae eq. (38), where in each recursion step a division with a quantity less than one ($s = R_b^2/(r \cdot r')$) takes place.

It was therefore decided to allow the use of the expression eq. (16) in high altitudes, but only carrying the summation up to some finite limit. The choice of summation limit and of the height in which this possibility should be used will depend on the numerical characteristics of the actual computer used.

The following procedure may be used to choose these limits:

(A) Compute the values of the covariances which make up the Laplace equation for all quantities given by eq. (1)– (14) in altitudes from e, g, 0 km to 1000 km in steps of 25 km using P identical to Q; (ψ=0). These values will show the magnitude of the error occurring while using the closed expressions.

The height in which the value of the Laplace equation exceeds e, g. the value obtained at the surface of the Earth or in which overflow occurs may then be chosen as the "critical height", h_{max} .

(B) Compute in the height some kilometers below h_{max} the same covariance quantities using the closed expressions and eq, (16) with varying summation limits. The summation limit may then be chosen by requiring that the difference between the values obtained using the closed expressions and the finite series is of the same numerical magnitude as the error observed when evaluating the Laplace equation using the closed expression.

The final version of COVAX (which is listed in the appendix) includes the possibility for the use of the finite series. The subroutine will therefore (besides the specification of the degree-variance model, etc.) require the specification of a logical variable LSUM, which is true in case eq. (16) will have to be used and false otherwise, It furthermore requires the specification of the value of h_{max} and of the summation limit,

The computations require in all cases the specification of three different: kinds of quantities

a. the radius of the Bjerhammar-sphere, R_b, the model degree-variances $\sigma_\ell^m(T,T)$, the degree-variance corrections $\sigma_\ell^c(T,T)$, (cf. eq. (16)), the values of LSUM, h_{max} and the summation limit.

b. the kind of quantities between which the covariances are to be computed, and

c. the coordinates of the points P and Q in which the quantities are evaluated and (in some cases) the reference gravity.

The subroutine has been designed accordingly, having three parts, each with a separate entry (COVAX, COVBX, COVCX). (A reader unfamiliar with terms such as "subroutine", "entry", etc. should consult e. g. IBM (1973, p. 96)).

The subroutine requires the specifications to be given in the following way (where all specifications labelled "a", "b", "c" must be done before the call of COVAX, COVBX, and COVCX, respectively).

(a-1) The degree-variance model is specified by giving the degree-variance model number (1, 2, or 3).

(a-2): The model degree-variances are specified by giving the value of A (cf. eq. (17)) in units of $(m/sec)^4$, of k_3 for model 2 and of k_2 and k_3 for model 3. The subroutine uses fixed values for k_0 (= -2) and k_1 (= -1). The choice of these quantities are in principle arbitrary, but the values have given good result in the analytic representation of empirical covariance functions, (cf. Tscherning and Rapp (1974, section 6)). The subroutine requres k_2 and k_3 to be positive (>0) when used. The lower summation limit p of eq. (20) has therefore been fixed to 3 in the subroutine.

(a-3): The radius of the Bjerhammar-sphere is specified by giving

$$s_0 = (R_E / R_b)^2,$$

where R_E has been chosen to 6371.0 km.

(a-4): For the degree-variance corrections there are three possibilities:

(I). A number of degree-variance corrections are used (maximal degree N). These are generally not known and will depend on the actual values specified in (a-1) - (a-3). What is known is on the other hand the empirical anomaly degree-variances at the surface of the Earth,

$$\sigma_\ell(\Delta g, \Delta g) \cdot s_0^{\ell+2}.$$

These will have to be transferred to COVAX together with the value of $N1 = N + 1$. The subroutine will compute the degree-variance corrections in units of $(m/sec)^4$ using

$$\sigma_\ell^c(T, T) = \frac{R_b^2 \cdot 10^{-10}\, \sigma_\ell(\Delta g, \Delta g)}{(\ell - 1)^2} - \sigma_\ell^m(T, T)$$

for $\ell > 2$ and with $\sigma_\ell(\Delta g, \Delta g)$ in units of $mgal^2$.

For $\ell = 2$, we simply use the same expression with the model degree-variance equal to zero. All terms of degree 0 and 1 are supposed to be equal to zero.

(II). No degree-variance corrections are used. This is indicated by assigning the variable N1 the value of the summation limit p (= 3).

(III). The degree-variance corrections of order up to an inclusive N are equal to minus the value of the model degree-variances. A representation of a local covariance function may be obtained in this way (cf. Ibid (1974, section 9)). A logical variable LOCAL is used to indicate that this possibility has been chosen. (It must have assigned the value true in this case and false in cases (I) and (II)).

(1): The kind of quantities between which the covariances is to be computed is specified by storing the values of the equation numbers defining the corresponding linear functionals (eq. (1)- (14)) in specific array elements, (cf. comment-statements included in COVAX).

(c-1): The coordinates of the points P and Q are specified indirectly by giving the sines and cosines to the latitude of P and Q and of their longitude difference $\Delta\lambda$.

(c-2): The reference gravities of P and Q must be given in units of m/sec^2.

How the explicit specifications actually are done are described in all detail in the subroutine itself through comment-statements. In order to clarify the use of the subroutine, the transfer of information and allocation of storage space for arrays a program calling the subroutine has been included in the appendix together with an in- and output example. A flowchart of the program is shown in Figure 1.

The program has been used for the computation of all covariance values in Table 6 and 7. The values given in the tables are shown with five digits behind the decimal point. This will not in all cases correspond to the actual number of correct digits, but they are shown in order to facilitate the comparison of results obtaining using different computers or FORTRAN compilers.

F should be noted, that the program uses a very simple expression for the reference gravity, namely

$$\gamma = GM/r^2.$$

For actual production type computations other expressions for the reference gravity should be considered, including expressions with and without the contribution from the rotation of the Earth.

We will finally .mention, that Algol-procedures, corresponding to COVAX may be obtained on request from the Danish Geodetic Institute.

Figure 1

Flow-chart of the calling program.

The following logical variables determines the flow:

LTEST = test-output is needed,

LASTl = \
LAST2 = } logical variables, true when quantities input simultaneously are the last specification, which will be input within the program-loop.
LAST 3 = /

F = false

T = true

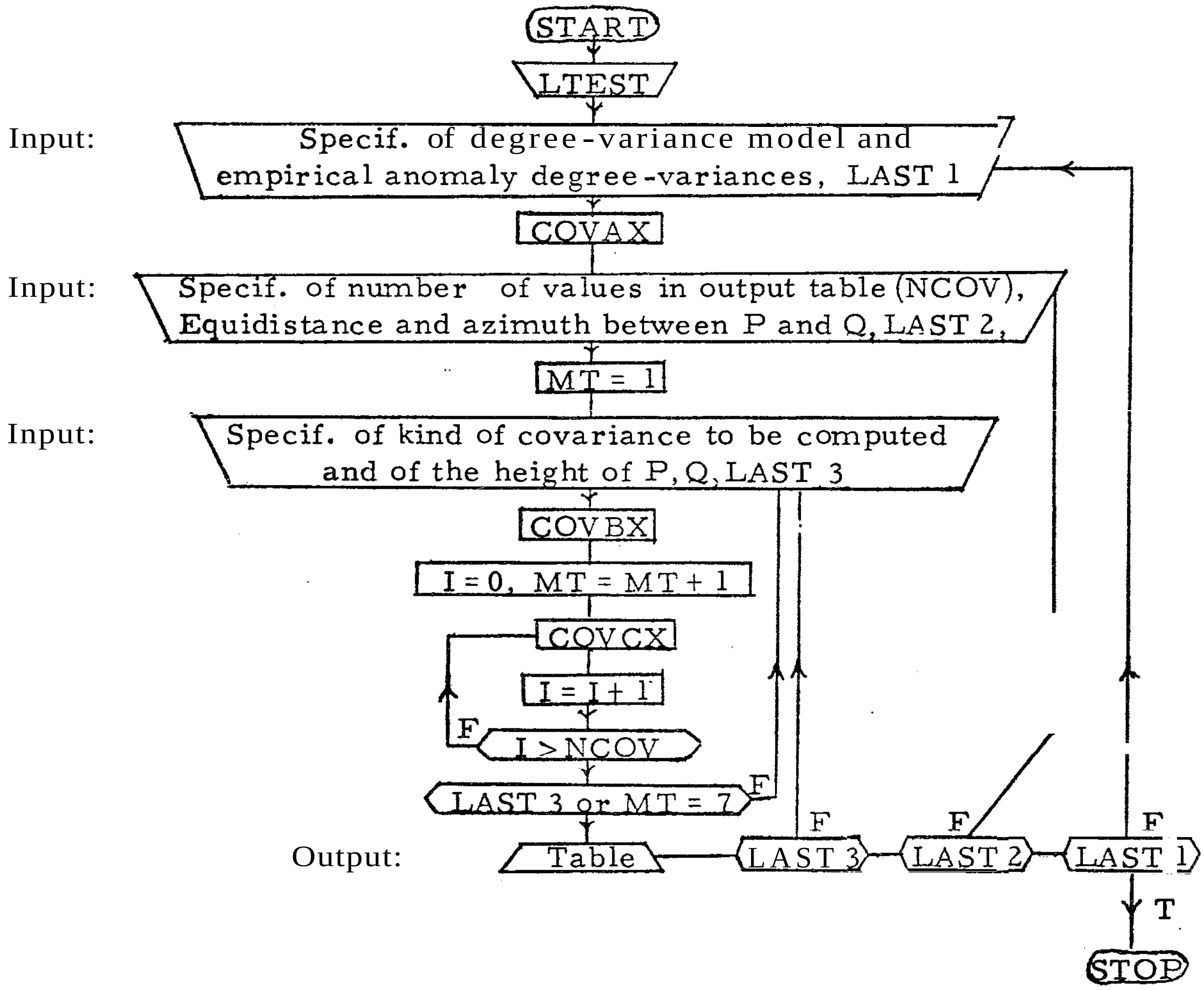

Table 6a. Table of covariances between quantities evaluated at points P and Q at the surface of the Earth, having spherical distance ψ and an azimuth of zero degrees from P to Q. The degree-variance model recommended by Tscherning and Rapp (1974, table 7) with $\sigma_2(\Delta g, \Delta g) = 7.5\ \mathrm{mgal}^2$ was used,

	Quantity given by equation:				
ψ ° '	(3) mgal^2	(4) E^2	(5) E^2	(8) E^2	(9) E^2
0 0.0	1795.00693	7084.59528	7095.24562	3538.33931	3538.33931
0 2.50	1604.93586	699.35412	704.19801	-342.98829	1038.77430
0 5.00	1418.83070	105.93168	108.49560	-253.38104	357.44824
0 7.50	1288.04626	29.15054	30.84237	-145.87968	173.81532
0 10.00	1189.51949	10.26853	11.51339	-91.97655	101.36198
0 12.50	1111.16205	3.87847	4.85436	-62.69912	65.89350
0 15.00	1046.46999	1.29380	2.09110	-45.28639	46.02779
0 17.50	991.60908	0.13473	0.80534	-34.15740	33.83282
0 20.00	944.14382	-0.41308	0.16331	-26.63622	25.83284
0 22.50	902.43525	-0.67351	-0.16976	-21.32494	20.31415
0 25.00	865.32950	-0.79004	-0.34388	-17.43957	16.35412
0 27.50	831.98432	-0.83151	-0.43202	-14.51405	13.42094
0 30.00	801.76657	-0.83299	-0.47202	-12.25770	11.19095
0 32.50	774.18859	-0.81316	-0.48449	-10.48183	9.45813
0 35.00	748.86690	-0.78233	-0.48108	-9.05966	8.08646
0 37.50	725.49469	-0.74630	-0.46860	-7.90356	6.98322
0 40.00	703.82267	-0.70839	-0.45110	-6.95138	6.08349
0 42.50	683.64565	-0.67047	-0.43103	-6.15807	5.34076
0 45.00	664.79275	-0.63362	-0.40990	-5.49035	4.72101
1 0.0	572.74180	-0.45151	-0.29280	-3.05700	2.51518
1 30.00	452.63643	-0.25007	-0.15212	-1.30715	1.00790
2 0.0	375.55500	-0.15418	-0.08416	-0.70128	0.51593
3 0.0	279.86457	-0.07151	-0.02698	-0.28202	0.19465
4 0.0	221.60304	-0.03885	-0.00591	-0.14387	0.09541
5 0.0	181.88207	-0.02324	0.00314	-0.08397	0.05428
10 0.0	86.14307	-0.00375	0.00976	-0.01450	0.00907
15 0.0	46.08241	-0.00133	0.00710	-0.00507	0.00316
25 0.0	8.35914	-0.00091	0.00190	-0.00150	0.00081
35 0.0	-8.54874	-0.00101	-0.00146	-0.00078	0.00027
45 0.0	-15.37408	-0.00098	-0.00323	-0.00050	0.00006
55 0.0	-15.89388	-0.00081	-0.00371	-0.00032	-0.00003
65 0.0	-12.44514	-0.00053	-0.00320	-0.00016	-0.00007
75 0.0	-6.94649	-0.00021	-0.00209	-0.00003	-0.00009
85 0.0	-1.05760	0.00009	-0.00075	0.00008	-0.00008
95 0.0	3.88042	0.00031	0.00049	-0.00014	0.00006
105 0.0	6.94307	0.00042	0.00137	-0.00018	0.00003
115 0.0	7.68960	0.00041	0.00173	-0.00019	-0.00001
125 0.0	6.17001	0.00029	0.00157	-0.00020	-0.00004
135 0.0	2.86437	0.00009	0.00096	-0.00019	-0.00008
145 0.0	-1.43769	-0.00016	0.00007	-0.00019	-0.00011
155 0.0	-5.79884	-0.00040	-0.00086	-0.00018	-0.00014
165 0.0	-9.31156	-0.00058	-0.00163	-0.00017	-0.00015
175 0.0	-11.26327	-0.00066	-0.00207	-0.00017	-0.00016
180 0.0	-11.51674	-0.00070	-0.00213	-0.00016	-0.00016

Table 6b. Table of covariances between quantities evaluated at points P and Q at the surface of the Earth, having spherical distance ψ and an azimuth of zero degrees from P to Q. The degree-variance model recommended by Tscherning and Rapp (1974, table 7) with $\sigma_2(\Delta g, \Delta g) = 7.5$ mgal^2 was used.

ψ °	ψ '	(10) E^2	(11) E^2	(12) E^2	(13) E^2	(14) E^2
0	0.0	3543.63376	3543.63376	2656.72945	885.57564	2656.72945
0	2.50	-341.51473	1042.08881	-314.94444	-26.94464	1068.20016
0	5.00	-252.93923	359.52007	-180.02387	-73.03198	432.02982
0	7.50	-145.70940	175.29046	-94.61752	-51.14018	226.05789
0	10.00	-91.90940	102.49624	-56.57669	-35.35476	137.56369
0	12.50	-62.67985	66.80900	-37.23202	-25.45750	92.03407
0	15.00	-45.29210	46.79162	-26.21984	-19.07535	65.67261
0	17.50	-34.17713	34.48563	-19.40097	-14.77544	49.09460
0	20.00	-26.66413	26.40103	-14.90241	-11.75871	38.01454
0	22.50	-21.35771	20.81581	-11.78556	-9.56772	30.25507
0	25.00	-17.47519	16.80218	-9.54047	-7.92940	24.61661
0	27.50	-14.55128	13.82496	-7.87158	-6.67381	20.39489
0	30.00	-12.29574	11.55815	-6.59833	-5.69117	17.15473
0	32.50	-10.52013	9.79415	-5.60550	-4.90819	14.61562
0	35.00	-9.09788	8.39575	-4.81684	-4.27451	12.59027
0	37.50	-7.94145	7.26934	-4.18028	-3.75462	10.94984
0	40.00	-6.98879	6.34938	-3.65930	-3.32296	9.60332
0	42.50	-6.19489	5.58883	-3.22771	-2.96071	8.48503
0	45.00	-5.52650	4.95327	-2.86630	-2.65382	7.54658
1	0.0	-3.08875	2.68089	-1.56591	-1.51718	4.15444
1	30.00	-1.33138	1.10911	-0.65255	-0.67458	1.75651
2	0.0	-0.72027	0.58645	-0.34345	-0.37360	0.94078
3	0.0	-0.29459	0.23655	-0.13399	-0.15864	0.38333
4	0.0	-0.15282	0.12419	-0.06670	-0.08485	0.20064
5	0.0	-0.09067	0.07576	-0.03815	-0.05167	0.12099
10	0.0	-0.01707	0.01780	-0.00637	-0.01053	0.02544
15	0.0	-0.00676	0.00834	-0.00250	-0.00414	0.01067
25	0.0	-0.00306	0.00334	-0.00129	-0.00145	0.00387
35	0.0	-0.00249	0.00163	-0.00113	-0.00090	0.00205
45	0.0	-0.00221	0.00074	-0.00100	-0.00072	0.00124
55	0.0	-0.00182	0.00021	-0.00079	-0.00065	0.00078
65	0.0	-0.00130	-0.00011	-0.00052	-0.00061	0.00047
75	0.0	-0.00070	-0.00029	-0.00021	-0.00059	0.00022
85	0.0	-0.00012	-0.00035	0.00007	-0.00070	-0.00013
95	0.0	-0.00029	0.00033	0.00017	-0.00025	-0.00028
105	0.0	-0.00053	0.00024	0.00015	-0.00006	-0.00016
115	0.0	-0.00068	0.00012	0.00012	-0.00004	-0.00018
125	0.0	-0.00074	-0.00003	0.00007	-0.00004	-0.00020
135	0.0	-0.00074	-0.00018	-0.00001	-0.00005	-0.00023
145	0.0	-0.00070	-0.00033	-0.00011	-0.00007	-0.00025
155	0.0	-0.00065	-0.00045	-0.00020	-0.00008	-0.00026
165	0.0	-0.00061	-0.00053	-0.00028	-0.00009	-0.00027
175	0.0	-0.00058	-0.00058	-0.00032	-0.00010	-0.00028
180	0.0	-0.00058	-0.00058	-0.00033	-0.00010	-0.00028

Column heading: Quantity given by equation:

5. Reflections over the choice of covariance function

In Tscherning and Rapp (1974) a specific covariance function is recommended (degree–variance model 2 of the present report, $(R_b/R_E)^2 = 0.999617$, $k_0 = -2$, $k_1 = -1$, $k_2 = 24$ and $A_1 / R_b{}^2 = 425.24$ mgal^2). Auto-covariance values for the quantities defined in eq. (3) - (5) and (8) - (14) are tabulated in Table 6a, b for varying spherical distance ψ.

The covariance function was chosen, so that it, in between several models, gave the best representation of different kinds of empirical free-air gravity anomaly data (a global 1° mean anomaly covariance function, the mean square variation of the point anomalies, etc.).

But will the covariance functions of other quantities be appropriately represented by expressions derived from the recommended model? Unfortunately, this can not be answered at present, because no globaly distributed samples of gravity dependent quantities other than gravity anomalies are available.

In Ibid. (section 9) it is explained, how a local covariance function may be represented by removing a number of the lower order degree-variances and by choosing a value for the constant A_1 (in eq. (17)) so that the empirically determined mean square variation of the local anomalies and the value derived from the 'model' covariance function becomes identical.

Such a local covariance function eq. (78a) was determined for the State of Ohio, U.S.A., (cf. Tscherning (1974, page 25)).

(78a) $$\operatorname{cov}(T_P, T_Q) = \sum_{\ell=205}^{\infty} \frac{(81.8\ \text{mgal}^2)\cdot R_b^2}{(\ell-1)(\ell-2)(\ell+24)}\, s^{\ell+1}\, P_\ell(\cos\psi),$$

$$(R_b/R_E)^2 = 0.9996$$

In Southern Ohio measurements of second order horizontal derivatives have been carried out in a little more than 300 points (Badekas (1967)). The following quantities were observed:

$(D^2_{\varphi r} W)/r$, $(D_{\lambda r} W)/(\cos\varphi\cdot r)$, $(D^2_\varphi W)/r^2 - (D^2_\lambda W)/(\cos^2\varphi\cdot r^2)$, and

$2\cdot(D^2_{\lambda\varphi} W)/(\cos\varphi\cdot r^2)$ or equivalently the corresponding derivatives (or linear combination of derivatives) of the anomalous potential, T.

Covariance functions between any of these latter quantities will be azimuth dependent, so they can not be used directly for the estimation of empirical covariance functions. But the two quantities,

$$(D^2_{\varphi r}\,T)/r \text{ and } (D^2_{\lambda r}\,T)/(r \cdot \cos\varphi)$$

may for each two points of observation be split into a longitudinal and a transversal component in the same manner as done for deflections of the vertical (cf. Tscherning and Rapp (1974, figure 2)). Doing this, it was possible to compute three local, empirical covariance functions, namely the auto-covariance functions of the longitudinal and transversal components and the cross-covariance function between the components. The covariance values were obtained (after first having subtracted the mean values from the two basic quantities separately) by computing the mean values of all products of quantities observed in points having a spherical distance falling within one of the sample intervals $(0 - \frac{1}{2}{}', \frac{1}{2}{}' - 1\frac{1}{2}{}', 1\frac{1}{2}{}' - 2\frac{1}{2}{}'$, etc.). The empirical values are shown in the figures 2a, b, and c.

Using the covariance function given by eq. (78a) corresponding covariance values were computed. They are shown in fig. 2a, b, c as well, and it appears that the values are significantly different. The 'model' mean square variation of the anomalies is much bigger than the empirical and the 'model' auto-covariance values have a much faster decrease than the empirical values.

We may then hope, that we by varying the parameters determining the 'model' covariance function will arrive to a model, which gives a better fit to the empirical data. A decrease in the radius of the Bjerhammar-sphere, for example, will reduce both the variation of the point gravity anomalies and the variation of the second order derivatives. The effect will be relatively bigger for the second order derivatives than for the gravity anomaly. Hence, by simultaneously increasing the constant A of eq. (17) and decreasing the radius of the Bjerhammar-sphere we may hold the mean square variation of the gravity anomalies fixed and arrive to a proper value for the mean square variation of the two second order derivatives we have regarded. Changing the ratio $(R_b/R_E)^2$ from 0.9996 to 0.9994 and A_a/R_b^2 from 82 to 100 $mgal^2$ we arrived to a model covariance function, which then at least has consistent values for the two mean square variations. Covariance values derived from this final model covariance function (78b) are shown in fig. 2a, b, and c as well.

$$\text{(78b)} \qquad \mathrm{cov}(T_P, T_Q) = \sum_{\ell=205}^{\infty} \frac{(100\ \mathrm{mgal}^2)\cdot(R_b^2)}{(\ell-1)(\ell-2)(\ell+24)} \cdot s^{\ell+1} P_\ell(\cos\psi).$$

Figure 2a, b, c. Local empirical and "model" covariance values for the longitudional and transversal components of the quantities given by eq. (8) and (9).

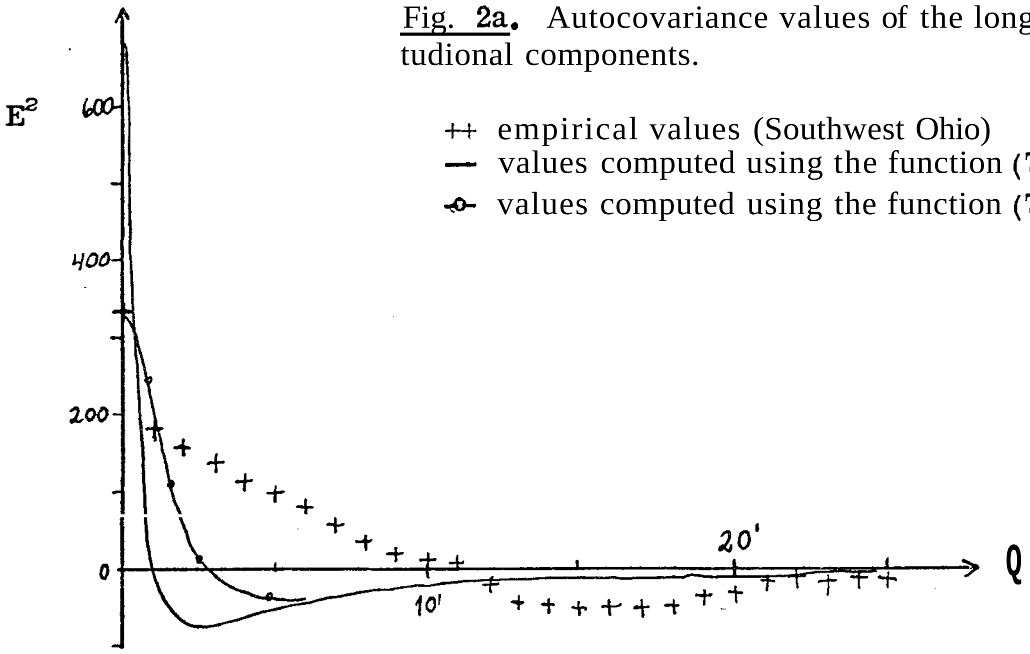

Fig. 2a. Autocovariance values of the longitudional components.

++ empirical values (Southwest Ohio)
— values computed using the function (78a)
-o- values computed using the function (78b).

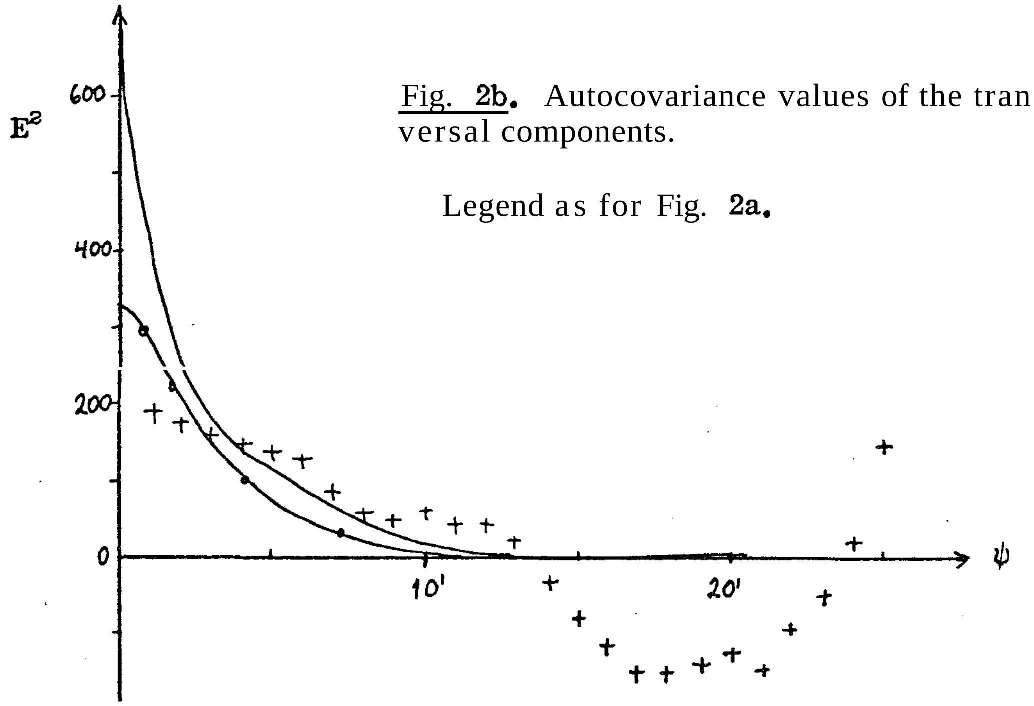

Fig. 2b. Autocovariance values of the transversal components.

Legend as for Fig. 2a.

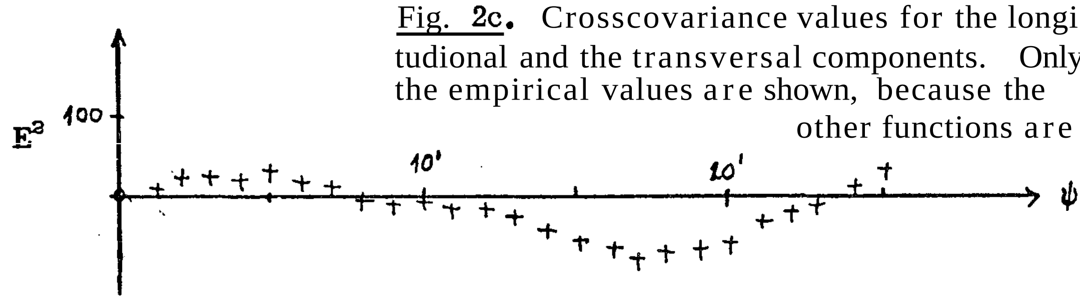

Fig. 2c. Crosscovariance values for the longitudional and the transversal components. Only the empirical values are shown, because the other functions are 0.

By describing this small investigation we wish to call attention to the complicated problem of deriving covariance functions, which sufficiently well represent the actual variations of the gravity field.

Let us finally regard three covariance functions, each using a different degree-variance model (eq. (17), i=1, 2, and 3), each having the same corresponding mean square variation of the point anomalies of 1795 $mgal^2$ and each giving a reasonable fit to the empirical degree-variances of order 3 to 20, (cf. Tscherning and Rapp (1974, Table 5)). The parameters which remain to be fixed are then for all three models the radius R_b (or equivalently the ratio $s = (R_b^2/R_E^2)$), for model 2 in addition the integer k_2 and for model 3 in addition the integers k_2 and k_3. By varying these parameters we then arrived to the values given in Table 7a (the covariance functions **A, B,** and D), (The values for model 2 are naturally identical to the values given in Ibid. (Table 7)).

In each model the contribution from degree ℓ to the variance of the anomalies is equal to

$$\frac{A_i}{R_b^2}(\ell-1)^2 \prod_{j=0}^{i} \frac{1}{(\ell+k_j)} \cdot \left(\frac{R_b}{R_E}\right)^{2\ell+4} . \tag{79}$$

In order to make the terms of degree greater than 20 add up to the same figure (1795 $mgal^2$ minus the sum of the degree-variances of orders less than and equal to 20 (Ibid. (Table 5))), the radius of the Bjerhammar-sphere will have to be the smallest for model 1 and the biggest for model 3 (as it also appears from Table 7a).

Table 7a. Quantities defining the covariance functions (A) - (D), (cf. eq. (16) and (17)), used to compute the mean square variations given in Table 7b.

Covariance function	Deg. var. model	$\frac{R_b^2}{R_E^2}$	A/R_b^2 $mgal^2$	k_2	k_3	n	$\sigma_2(\Delta g, \Delta g)$ $mgal^2$
A	1	0.996004	7.2			2	7.5
B	2	0.999617	425.3	24		2	7.5
C	2	0.999617	425.3	24		20	
D*	3	0.9999	465110.0	13	1100	2	7.5

* Model D used with LSUM true, ℓ_{max} = 25km and the summation limit equal to 300.

Table 7b. Mean square variations of different quantities for varying height computed using four different covariance functions, defined by the constants given in Table 7a.

Height km	Cov. fct.	(1) m^2	(3) $mgal^2$	(4) E^2	(6) $(arc\ sec)^2$
		Quantity given by equation:			
0	A	606.26265	1795.07333	548.37873	43.42127
0	B	926.59371	1795.00693	7084.59528	45.30758
0	C	13.31572	1519.62082	7084.48215	34.53820
0	D	1304.48044	1795.00293	8995.06133	47.65294
10	A	600.44125	1008.53709	96.53372	25.86525
10	B	917.64976	931.90126	79.15527	26.06286
10	C	11.73505	666.43502	79.04804	15.55471
10	D	1291.66832	1090.89666	64.79279	31.91158
100	A	559.45831	200.60616	0.79982	7.23262
100	B	849.50341	295.08825	0.70300	11.14269
100	C	4.73447	101.41301	0.63592	2.60397
100	D	1191.82157	408.69105	0.79292	15.76414
250	A	508.34093	79.81504	0.06446	3.99956
250	B	760.73519	138.99049	0.08271	6.84944
250	C	1.36335	16.10806	0.05031	0.51966
250	D	1059.75459	204.59633	0.10628	10.07583
500	A	443.97868	34.38415	0.00894	2.46515
500	B	648.85476	64.08980	0.01458	4.29061
500	C	0.22216	1.91983	0.00341	0.06475
500	D	892.75725	97.46851	0.02061	6.38635

Table 7b. (cont)

Height km	Cov. fct.	Quantity given by equation: (8) E^2	(10) E^2	(12) E^2	(13) E^2
0	A	272.54346	274.75402	205.64801	68.54853
0	B	3538.33931	3543.63376	2656.72945	885.57564
0	C	3538.29401	3543.56958	2656.68560	885.56123
0	D	4494.26797	4498.63571	3373.15391	1124.38375
10	A	47.75032	48.44846	36.20366	12.06741
10	B	39.20790	39.71032	29.68674	9.89510
10	C	39.16503	39.64940	29.64516	9.88143
10	D	32.05089	32.52310	24.30158	8.09994
100	A	0.37894	0.40927	0.30081	0.10015
100	B	0.32826	0.36290	0.26497	0.08814
100	C	0.30180	0.32423	0.23880	0.07956
100	D	0.36772	0.41121	0.29925	0.09948
250	A	0.02842	0.03462	0.02458	0.00813
250	B	0.03559	0.04520	0.03172	0.01047
250	C	0.02314	0.02598	0.01892	0.00630
250	D	0.04519	0.05864	0.04090	0.01349
500	A	0.00353	0.00533	0.00355	0.00115
500	B	0.00562	0.00881	0.00582	0.00189
500	C	0.00154	0.00178	0.00129	0.00043
500	D	0.00783	0.01256	0.00825	0.00267

Without considering the magnitude of the radius of the Bjerhammar-sphere, one would conclude, based on the difference in the behaviour of the degree-variances for increasing degree, that the mean square variation of the second order derivatives would be much smaller using model 3 than model 1. However, from Table 7b and 7c it appears, that this is not the case. The mean square variation of the second order derivatives derived using the covariance function labeled **A**, (and which uses degree-variance model 1), has the smallest values. This is due to the dampening effect of the quantity $(R_b/R_E)^2$ in eq. (79).

We have furthermore in Table 7b and 7c tabulated the values of the mean square variations for different second order derivatives at different heights. Note, that for high altitudes (where only the low order harmonics have an effect),, the three models give approximately the same values. The values derived from another covariance function (labeled C) can also be found in the tables. This covariance function is a local 20'th order covariance function corresponding to the global covariance function B. It is, for this local covariance function, interesting to see how little the mean square variations of the second order derivatives differ from the values derived using the covariance function B.

(We will in this connection call attention to the investigations of the height variations presented in Reed (1973, section 4) which are based on a degree-variance model similar to model 2 of this report).

We have seen in this section how differently the four different covariance functions may represent the variations of the gravity field at the surface of the Earth and anyway be similar at high altitudes. Hence, for some purposes we may be quite uncritical in our choice of covariance function, and for some purposes we may discover, that we are not able to find an appropriate model.

6. Conclusion

In this report we have derived covariance functions of second and lower order derivatives of the anomalous potential and a FORTRAN IV subroutine for their numerical evaluation is documented. The knowledge of these covariance functions is a necessity for many geodetic applications of these derivatives.

The covariance functions are given through the specification of different properties and parameters (rotational invariance, the behaviour of the degree-variances when the degree goes to infinity, the radius of the Bjerhammar-sphere, etc.). It is hoped, that it, in between the here discussed set of covariance functions, may be possible to find global and local models representing the actual variations of the gravity field. For the further study of this point, observations of second order derivatives must be carried out in regions with different geological and topographical conditions.

However, a covariance function (of equivalently a norm in a Hilbert space of harmonic functions), may be chosen only of numerical reasons. Hence, a covariance functions may be useful even when it does not represent the actual variations of the gravity field. It usefulness will depend on e.g. the quality of the predictions obtained using the function.

References

Badekas, J.: Torsion Balance Observations in Southwest Ohio. - Report of the Department of Geodetic Science, Report, No. 89, The Ohio State University, Columbus, 1967.

(IBM): IBM System/360 and System/370 FORTRAN IV Language, GC28-6515-9, 1973.

Krarup, T.: A Contribution to the Mathematical Foundation of Physical Geodesy, The Danish Geodetic Institute, Meddelelse No. 44, 1969.

Moritz, Helmut: Some First Accuracy Estimates for Applications of Aerial Gradiometry. - Report of the Department of Geodetic Science, Report No. 209, The Ohio State University, Columbus, 1974.

Reed, George B.: Application of Kinematical Geodesv for Determining the Short Wave Length Components of the Gravity Field by Satellite Gradiornetry. - Reports of the Department of Geodetic Science, Report No. 201, The Ohio State University, Columbus, 1973.

Tscherning, C. C: On the Relation Between the Variation of the Degree-Variances and the Variation of the Anomalous Potential, Bollettino di Geodesia e Scienze Affini, Anno XXXII, N. 3, 1973.

Tscherning, C.C.: A FORTRAN IV Program for the Determination of the Anomalous Potential Using Stepwise Least, Squares Collocation. - Reports of the Department of Geodetic Science, Report No. 212, The Ohio State University, Columbus, 1974.

Tscherning, C. C. and Rapp, Richard H.: Closed Covariance Expressions for Gravity Anomalies, Geoid Undulations, and Deflections of the Vertical Implied by Anomaly Degree-Variance Models. - Reports of the Department of Geodetic Science, Report No. 208, The Ohio State University, Columbus, O., 1974.

Appendix

A. The FORTRAN IV subroutine COVAX.

B. An example of a program calling the subroutine, with a input and corresponding output example.

```
C TEST OF COVARIANCE FUNCTION SUBROUTINES, PROGRAMMED BY C.C.TSCHERNING,
C DEP.GEODETIC SCIENCE, OSU AND GEODAETISK INSTITUT,KOEBENHAVN JUNE 75.
C REFERENCES:
C (A) TSCHERNING,C.C. : COVARIANCE EXPKESSIONS FOK SECOND AND LOWER ORDER
C      DERIVATIVES OF THE ANOMALOUS POTENTIAL, REPORTS OF THE DEP. OF
C      GEODETIC SCIENCE NO. 225,1975.
C
      IMPLICIT REAL *8(A-H,O-Z),LOGICAL (L)
      COMMON /CMCOV/CI(12),CR(51),SIGMAO(300),SIGMA(300),HMAX,KI(25),N1,
     *N2,LOCAL, LSUM
C THE COMMON AREA IS USED FOR THE TRANSFER OF DATA TO AND FROM THE SUB-
C ROUTINE COVAX.
      DIMENSION COV(181,7),KP(7),KQ(7),HP(7),HQ(7)
     *,KX(3),KY(3),IP(3),IQ(3),LA(3),SM(2001)
      DATA GM, RE/3.98D14,6371.0D3/
     *,DO,D1,D2/0.0D0,1.0D0,2.0D0/,PI/3.1415926535D0/
C RE IS THE MEAN RADIUS OF THE EARTH AND GM IS THE PRODUCT OF THE GRAVI-
C TATIONAL CONSTANT AND THE MASS OF THE EARTH.
C
      WRITE(6,10)
   10 FORMAT('OTEST OF COVARIANCE FUNCTION SUBROUTINES, VERS. JUNE 75.',
     *//,' COVARIANCES BETWEEN QUANTITIES OF KIND KP,KQ ARE COMPUTED. ',
     */,' THE KINDS AND CORRESPONDING UNITS ARE AS FOLLOWS: (E=EOTVOS'
     *,'):',/,
     *' (1) THE HEIGHT ANOMALY (METERS), (2) THE NEGATIVE RADIAL DER-',/
     *,' IVATIVE DIVIDED BY THE RADIAL DISTANCE (E), (3) THE GRAVITY',/,
     *' ANOMALY (MGAL), (4) THE RADIAL DERIVATIVE OF (3) (E), (5) THE'/
     *' SECOND ORDER RADIAL DERIVATIVE (E), (6),(7) THE LATITUDE AND',/,
     *' THE LONGITUDE COMPONENTS OF THE DEFLECTIONS OF THE VERTICAL',/,
     *' (ARCSECONDS), (8),(9) THE DERIVATIVES OF (3) IN NORTHERN AND',/,
     *' EASTERN DIRECTION, RESPECTIVELY (E), (10),(11) THE DERIVATIVE',/
     *,' OF (2) IN THE SAME DIRECTIONS (E), (12)-(14) THE SECOND ORDER',
     */,' DERIVATIVES IN NORTHERN, (NORTHERN, EASTERN) AND EASTERN',/,
     *' DIRECTIONS, RESPECTIVELY (E).',//)
C
C INPUT OF THE VALUE OF A LOGICAL VARIABLE, LTEST, TRUE WHEN TEST OUT-
C PUT IS NEEDED.
      READ(5,11)LTEST
   11 FORMAT(L2)
C INPUT OF QUANTITIES SPECIFYING THE DEGREE- VARIANCE MODEL TO BE USED IN
C THE FOLLOWING SEQUENCE: THE RATIO BETWEEN AN ADOPTED BJERHAMMAR-SPHERE
C RADIUS (RB) AND RE, SQUARED, THE QUANTITY A(I) IN REF(A),EQ.(17), DI-
C VIDED BY RB**2 IN UNITS OF MGAL**2, THE INTEGEKS K(2),K(3) OF EQ.
C (17), WHEN APPLICABLE* OTHERWISE A ZERO, THE VALUE OF A LOGICAL VARI-
C ABLE* LOCAL, TRUE,WHEN THE DEGREE-VARIANCES UP TO-AND INCLUSIVE DEGREE
C N ARE ZERO AND FALSE WHEN EMPIRICAL ANOMALY DEGREE-VARIANCES UP TO
C ORDER N WILL BE INPUT, THE INTEGER N, THE INTEGER KT EQUAL TO THE
C DEGREE- VARIANCE MODEL NUMBER (1,2 OR 3), THE VALUE OF THE LOGICAL
C VARIABLE LSUM, WHICH IS TRUE WHEN A FINITE LEGENDRE SERIES (MAXIMAL
C DEGREE 2000) MUST BE USED FOR THE EVALUATION OF COVARIANCES IN ALTI-
C TUDES GREATHER THAN HMAX AND OTHERWISE EQUAL TO FALSE, THE VALUE OF
C N2 (WHICH MUST BE LESS THAN OR EQUAL TO 2000), THE VALUE OF THE HEIGHT
C HMAX IN METERS, AND FINALLY THE VALUE OF LAST1, TRUE, WHEN THE LIST
C OF INPUT PARAMETERS IS THE LAST ONE.
C
  100 READ(5,9)S,A,KI(3),KI(4),LOCAL,N,KT,LSUM,N2,HMAX,LAST1
    9 FORMAT(2D14.7,2I5,L2,2I5,L2,I5,D14.7,L2)
      IF (N2.LT.2) N2 = 2
      IF (N2.GT.2001) N2 = 2000
      N2 = N2+1
      KI(5) = KT
      WRITE(6,12)S,A,KI(3),KI(4),N,KT
```

```
   12 FORMAT('OPARAMETERS SPECIFYING THE MODEL DEGREE-VARIANCES:',/,
     *' S,A =',2D14.7,/,' K0-K3,N,KT=   -2   -1',4I5/)
      IF (LSUM) WRITE(6,6)HMAX,N2
    6 FORMAT(' WHEN THE HEIGHT OF ONE OF THE POINTS OF EVALUATION IS ABO
     *VE ',D14.7,' METERS',/,' WILL THE COVARIANCES BE EVALUATED BY MEAN
     *S OF A LEGENDRE SERIES HAVING ',I5,' TERMS.')
      RB2 = RE*RE*S
      LMODEL = N.EQ.0
C
      CI(8) = A*RB2*1.0D-10
      CI(10) = S
C CONVERTING THE CONSTANT A INTO UNITS OF (M/SEC)**4 AND SUBSEQUENT
C STORING OF A AND S IN THE ARRPY CI ACCORDING TO THE SPECIFICATIONS
C GIVEN IN COVAX.
C
      IF (N.NE.0) GO TO 101
C N EQUAL TO ZERO IMPLIES, THAT ALL DEGREE-VARIANCES ARE EQUAL TO THE
C MODEL DEGREE-VARIANCES. THIS AGAIN IS EQUIVALENT TO HAVING THE DEGREE-
C VARIANCES OF DEGREE 0,1,2 EQUAL TO ZERO, I.E.  THE CUVARIANCE FUNCTION
C USED IS A 2'-ORDER LOCAL COVAKIANCE FUNCTION.
      LOCAL = .TRUE.
      N = 2
C
  101 N1 = N+1
C INPUT OF EMPIRICAL ANOMALY DEGRFE-VARIANCES IN UNITS OF MGAL**2.
      IF(.NOT.LOCAL)READ(5,13)(SIGMAO(I), I = 11 N1)
   13 FORMAT(12F6.2)
      IF (.NOT.LOCAL)WRITE(6,7)(SIGMAO(I), I = 1, N1)
    7 FORMAT(' EMPIRICAL ANOMALY DEGREE-VARIANCES IN UNITS OF MGAL**2:',
     */,25(12F6.2/))
C
      N2 = 2001
      CALL COVAX(SM)
C THE ARRAY SM OCCIJRRTNG IN THE CALL OF COVAX IS USED TO STORE THE
C DEGREE-VARIANCES WHEN LSUM IS TRUE.  THE DIMENSION OF THE ARRAY IS
C TRANSFERRED TO THE SUBROUTINE BY MEANS OF THE VARIABLE N2 OCCURRING
C IN THE COMMON AREA /CMCOV/.
C
C CONVERSION OF ANOMALY DEGREE-VARIANCES TO POTENTIAL-DEGREE-VARIANCES.
      WRITE(6,8)(SIGMAO(I), I = 1, N1)
    8 FORMAT('OMODEL DEGREE-VARIANCE CORRECTIONS:',/,50(6(1X,D11.4),/))
C
      KPP=0
      KQQ=0
C INPUT OF QUANTITIES SPECIFYING THE OUTPUT TABLE. THE TABLE WILL CON-
C TAIN MT COLUMN? OF COVARIANCES OF KINDS KP, KQ (TO BE INPUT SUBSEQUE-
C NTLY) COMPUTED IN POINTS P AND Q. EACH COLUMN WILL CONTAIN NCOV VALUES
C CORRESPONDING TO Q MOVING IN AN AZIMUTH (ALFA) IN STEPS OF LENGTH DT
C (MINUTES).  THE SPECIFICATION CONSIST OF NCOV, DT, ALFA IN DEG.,MIN.,
C SEC AND A LOGICAL, LAST2, WHICH IS TRUE WHEN THIS IS THE LAST SPECI-
C FICATION FOR THE DEGREE-VARIANCE MODEL UNDER CONSIDEKATION.
  102 READ(5,14)NCOV,DT,IDEG,MIN,SEC,LAST2
   14 FORMAT(I5,F7.2,I5,I3,F6.2,L2)
      CALL RAD(ALFA,IDEG,MIN,SEC,1)
      RT = 60.0*DT/206264.806
      LPOLE = DABS(ALFA).LT.1.0D-6.OR.DABS(ALFA-PI).LT.1.0D-6
      IF (LPOLE) GO TO 103
C
      CA = DCOS(ALFA)
      SA = DSIN(ALFA)
  103 IF (LTEST) WRITE(6,15)NCOV,DT,IDEG,MIN,SEC,ALFA
   15 FORMAT('ONCOV,DT,DEG,MIN,SEC,ALFA=',I4,F6.2,I5,I3,F6.2,D14.6)
```

```
C
      MV = 0
      MT = 1
C
      IF (NCOV.LE.181) GO TO 104
      NCOV = 181
      WRITE(6,37)
   37 FORMAT(' NCOV TOO BIG, FIXED TO 181.')
  104 MT = MT+1
      IF (MV.NE.0.AND.MV.NE.3) GO TO 112
C INPUT OF INTEGERS KP AND K8 SIGNIFYING THE KIND OF QUANTITIES BETWEEN
C WHICH WE WANT TO COMPUTE THE COVARIANCES. THE VALUES OF KP,KQ MUST BE
C EQUAL TO THE EQUATION NUMBERS OF REF(A), WHICH DEFINES THE QUANTITIES
C (1) - (14).
C ON THE SAME PUNCH CARD INPUT OF THE HEIGHTS OF P AND Q AS WELL,(IN
C METERS),AND OF A LOGICAL VARIABLE LAST3, TRUE WHEN THIS KIND OF COVA-
C RIANCES ARE THE LAST ONES TO BE COMPUTED WITH THE CHOOSED FORM OF THE
C TABLE. THREE SETS OF VALUES MAY BE PUNCHED ON ONE CARD, CF. FORMAT'
C STATEMENT 15.
      READ(5,16)(KX(K),KY(K),IP(K),IQ(K),LA(K),K = ı r 3)
   16 FORMAT(3(2I3,2I8,L2))
      MV = 0
  112 MV = MV+1
      KP(MT) = KX(MV)
      KQ(MT) = KY(MV)
      HP(MT) = IP(MV)
      HQ(MT) = IQ(MV)
      LAST3 = LA(MV)
C
      KI(6) = KP(MT)
      KI(7) = KQ(MT)
      LNEW = KP(MT).NE.KPP.OR.KQ(MT).NE.KQQ
C COMPUTATION OF CONSTANTS NEEDED FOR THE COVARIANCE COMPUTATIONr WHICH
C ARE INDEPENDENT OF T AND THE HEIGHTS BY THE CALL OF COVBX.
      IF (LNEW) CALL COVBX
      KPP = KP(MT)
      KQQ = KQ(MT)
C
      IF (LTEST. AND. LNEW)
     *WRITE(6,17)(CI(K),K=1,7),(KI(K),K=6,25),(SIGMA(K),K=1,N1)
   17 FORMAT('0CI:',7D11.4,/,' KI:',20I3,/,' SI:',5D11.4,/,59(4X,5D11.4/
     *))
C
      DO 120 M = ı r NCOV
      IF (MT.EQ.2) COV(M,1) = (M-1)*DT
      RV = (M-1)*RT
      T = DCOS(RV)
C RV IS EQUAL TO THE SPHERICAL DISTANCE BETWEEN P AND Q IN UNITS OF RAD-
C IANS.
      U = DSIN(RV)
      IF (LPOLE) GO TO 105
      SO = U*CA
      CO = DSQRT(D1-SO *SO)
      SD = U*SA/CO
      CD = T/CO
    . G0 TO 106
C
  105 SD = DO'
      CD = D1
      IF (RV.GT.PI/D2) CD = -D1
      CO = T
      I = 1
```

```
      IF (ALFA.LT.DO) I = -1
      SO = U* I
C
C TRANSFER OF COORDINATE INFORMATION TO THE SUBROUTINE ACCORDING TO
C THE SPECIFICATIONS GIVEN IN THE SUBROUTINE.
  106 CR(1) = T
      CR(2) = HP(MT)
      CR(3) = HQ(MT)
      CR(4) = DO
      CR(5) = SQ
      CR(6) = D1
      CR(7) = CQ
      CR(8) = SD
      CR(9) = CD
      CR(10) = GM/(RE+HP(MT))**2
      CR(11) = GM/(RE+HQ(MT))**2
      IF (LTEST)WRITE(6,18)SQ,CQ,SD,CD
   18 FORMAT('0SQ,CQ,SD,CD=',4D12.5)
      CALL COVCX(COV(M,MT))
C
      IF (.NOT.LTEST) GO TO 120
      KK = KI(8)+1
      WRITE(6,19)((CR(I*8+K+3),K=1,8),I=1,KK)
   19 FORMAT(' CR:',8D11.4,/,4(4X,8D11.4,/))
  120 CONTINUE
C
      IF (.NOT.(LAST3.OR.MT.EQ.7)) GO TO 104
C
C OUTPUT OF A TABLE OF COVARIANCES.
      WRITE(7,30)
      WRITE(6,30)
   30 FORMAT(' ')
      WRITE(6,20)IDEG,MIN,SEC
      WRITE(7,20)IDEG,MIN,SEC
   20 FORMAT( ' TARLE OF COVARIANCES:',/,
     *' BETWEEN QUANTITIES OF KIND KP AND KQ, EVALUATED IN P,Q,',/,
     *' HAVING SPHERICAL DISTANCE PSI, HEIGHTS HP, HQ',/,
     *' AND AN AZIMUTH OF',I5,' D',I3,' M',F6.2,' SEC FROM P TO Q.')
      WRITE(6,30)
      WRITE(7,30)
      WRITE(6,21)(KP(I),I=2,MT)
      WRITE(7,21)(KP(I),I=2,MT)
   21 FORMAT('      KP=   ',6(I6,5X))
      WRITE(6,22)(KQ(I),I=2,MT)
      WRITE(7,22)(KQ(I),I=2,MT)
   22 FORMAT('      KQ=   ',6(I6,5X))
      WRITE(6,23)(HP(I),I=2,MT)
      WRITE(7,23)(HP(I),I=2,MT)
   23 FORMAT('      HP=   ',6(1X,F10.1))
      WRITE(6,24)(HQ(I),I=2,MT)
      WRITE(7,24)(HQ(I),I=2,MT)
   24 FORMAT('      HQ=   ',6(1X,F10.1))
      WRITE(6,26)
      WRITE(7,26)
   26 FORMAT(' PSI')
C
      DO 113 K = 1, NCOV
      RM = COV(K,1)
      IE = IDINT(RM/60.0D0)
      RM = RM-60.0D0*IE
      WRITE(7,25)IE,RM,(COV(K,I), I = 29 MT)
  113 WRITE(6,25)IE,RM,(COV(K,I), I = 29 MT)
```

```
   25 FORMAT(I4,F6.2,6F11.5)
C
      MT = 1
      IF (.NOT.LAST3) GO TO 104
C
      IF (.NOT.LAST2) GO TO 102
C
      IF (.NOT.LAST1) GO TO 100
      STOP
      END
      SUBROUTINE RAD(RA,IDEG,MIN,SEC,MODE)
      IMPLICIT REAL*8(A-H,O-Z)
      DATA RS,PI/206264.806D0,3.1415926535D0/
      IG = 1
      IF (IDEG.LE.0 .AND.  MODE.EQ.1) IG = -1
      IDEG = IDEG*IG
      IF (MODE.NE.1  .OR. MIN .GE. 0) GO TO 20
      IG = -1
      MIN = MIN*IG
C
   20 GO TO (30,40,50,60),MODE
   30 RA =IDEG*3600.0D0+MIN*60.0D0+SEC
      GO TO 70
   40 RA =IDEG*3600.0D0+SEC*60.0D0
      GO TO 70
   50 RA = SEC*3600.0D0
      GO TO 70
   60 RA = SEC*3240.0D0
   70 RA = RA/RS
   80 IF (DABS(RA).LE.PI) GO TO 90
      RA = RA-2.0D0*PI*DSIGN(RA,1.0D0)
      GO TO 80
   90 RETURN
      END
      SUBROUTINE COVAX(SM)
C THE SUBROUTINE COMPUTES THE COVARIANCE BETWEEN TWO QUANTITIES OF A
C KIND SPECIPIED THROUGH THE VALUE OF TWO INTEGER VARIARLES (STORED IN
C KI(6) AND KI(7), SEE BELOW), THE QUANTITIES ARE EVALUATED IN TWO
C POINTS* P AND Q, THE COORDINATES OF WHICH ARE GIVEN IMPLICITLY BY THE
C VALUES OF CR(1) - CR(9).
C
C THE COVARIANCE FUNCTION USED IS DEFINED ACCORDING TO A DEGREE-VARIANCE
C MODEL AND A SET OF EMPIRICAL (POTENTIAL) DEGREE-VAHIANCES. THE UEGREE-
C VARIANCE MODEL IS SPECIFIED THROUGH THE VALUES OF KI(1)-KI(5),CI(8)-
C CI(10) AND THE PARAMETERS N1 AND LOCAL OCCURRING IN THE COMMON BLOCK
C /CMCOV/. EMPIRICAL ANOMALY DEGREE-VARIANCES WILL HAVE TO BE STORED IN
C SIGMAO WHEN LOCAL IS FALSE. AND ARE USED FOR THE COMFUTATION OF RESI-
C DUAL POTENTIAL DEGREE-VARIANCES, (SEE REF(A), EQ.(16)).
C
C THE SUBROUTINE HAS THREE ENTRIES, COVAX, COVRX AND COVCX, WHICH HAVE
C TO BE CALLFD IN THIS SEQUENCE.
C
C BY THE CALL OF COVAX, THE KIND OF COVARIANCE FUNCTION TO RE USED IS
C DETERMINED. THE VALUE OF KI(5) WILL DETEKMINE THE DEGREE-VARI-
C ANCE MODEL (1,2 OH 3, CF.REF(A),EQ.(17)) THAT WILL BE USED. THE OUAN-
C TITJES K(2),K(3) MUST HE STORED IN KI(3),KI(4), AND RE EOUAL TO ZERO
C WHEN NOT USED (EG.,KI(3),KI(4) BOTH ZERO WHEN KI(5)=1). THE QUANTITY
C A(I) MUST RE STOKED JN CI(8) IN UNITS OF (M/SEC)**4, AND THE SQUARE OF
C THE RATIO BETWEEN THE RADIUS OF THE BJERHAMMAR-SPHERE (RB) AND THE
C MEAN RADIUS OF THE EARTH (RE) MUST RE STORED IN CI(10).
C
C THERE ARE THEN THREE POSSIBILITIES:
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C (1) ONE OF THE DEGREE-VARIANCE MODELS IS USED WITHOUT MODIFICATIONS.
C     THE SUMMATION LIMIT P OF REF.(A),EQ.(20) IS THEN FIXED TO 3.
C     BECAUSE THIS IS EQUIVALENT TO REQUIRING THE FIRST 3 DEGREE-VARIAN-
C     AREA /CMCOV/ MUST BE EQUAL TO 3 AND .TRUE., RESPECTIVELY.
C     CES TO RE ZERO, THE VARIABLES N1 AND LOCAL STORED IN THE COMMON
C (2) A NUMBER (N1) OF THE ANOMALY DEGREE-VARIANCES (DEGREE ZERO TO
C     NI-11 ARE PUT EQUAL TO EMPIRICAL DETERMINED QUANTITIES. THE ANO-
C     MALY DEGREE-VARIANCE OF DEGREE K WILL HAVE TO BE STORED IN
C     SIGMAO(K+1) IN UNITS OF MGAL**2 WHEN CALLING COVAX. LOCAL MUST BE
C     EQUAL TO .FALSE..  COVAX WILL CONVERT THE ANOMALY DEGRE5-VARIANCES
C     INTO POTENTIAL DEGREE-VARIANCES.
C (3) THE N1 FIRST DEGREE-VARIANCES (DEGREE 0 - N1-1) ARE EQUAL TO ZERO,
C     THIS MEANS* THAT THE VALUES OF A (N1-1)-ORDER LOCAL COVARIANCE
C     FUNCTION WILL RE COMPUTED. LOCAL MUST HAVE THE VALUE .TRUE..
C IN ALL CASES N1 MUST BE LESS THAN 300,
C
C THE COVARIANCES WILL GENERALLY BE COMPUTED BY CLOSED EXPRESSIONS* BUT
C THEY MAY IN CERTAIN CASES RE USELESS IN BIG ALTITUDES OF NUMERICAL
C REASONS, CF. REF(A), SECTION 4. IN THEESE CASES MUST THE LUGICAL VARI-
C ABLE LSUM BE TRUE AND THE VARIABLE HMAX MUST HAVE ASSIGNED A VALUE
C EQUAL TO THE CRITICAL ALTITUDE. WHEN LSUM IS TRUE AND THE HEIGHT OF
C P OR Q IS GREATHER THAN HMAX, WILL THE SERIES REF(A), EQ.(16), ABBRE-
C VIATED TO DEGREE N2-1 BE USED FUR THE COMPUTATION OF THE COVARIANCES,
C THE VALUES OF LSUM, N2 AND HMAX WILL (IN THE SAME HAY AS FOR THE PARA-
C METERS SPECIFYING THE DEGREE-VARIANCE MODEL) BE TRANSFEKRED TO COVAX
C THROUGH THE COMMON AREA /CMCOV/, BUT AN ARRAY SM IS TRANSFERRED AS A
C PARAMETER IN THE CALL IN ORDER TO ENAHLE VARIABLE DIMENSIONING (SPECI-
C FIED BY THE VARIABLE N2 IN /CMCOV/).
C
C THE CALL OF COVAX WILL ALSO INITIALIZE CERTAIN VARIABLES USED IN
C SUBSEQUENT COMPIJTATIONS.
C
C THE CALL OF COVBX WILL FIX CERTAIN CONSTANTS USED FOR THE COMPUTA-
C TIONS, WHICH ARE INDEPENDENT OF THE POINTS P AND Q. WHEN COVBX IS CAL-,
C LED, THE KIND OF QUANTITIES BETWEEN WHICH THE COVARIANCE IS TO BE
C COMPUTED MUST BE SPECIFIED.  THIS IS DONE BY STORING IN KI(6) AND
C KI(7) INTEGERS EQUAL TO THE EQUATION NUMBERS OF REF.A, EQ.(1) - (14)
C DEFINING THE QUANTITIES.
C
C THE CALL OF COVCX WILL RESULT IN THE COMPUTATION OF THE COVARIANCE ,
C WHICH IS TRANSFERRED TO THE CALLING PROGRAM THROUGH THE VARIABLE COV,
C INFORMATION RELATFD TO THE COORDINATES OF P AND Q MUST BE STORED IN
C THE ARRAY CR WHEN COVCX IS CALLED, SEE BELOW.
C REFERENCES:
C (A) TSCHERNING,C.C.: COVARIANCE EXPRESSIONS FOK SECOND AND LOWER ORDER
C     DERIVATIVES OF THE ANOMALOUS POTENTIAL* REPORTS OF THE DEP. OF
C     GEODETIC SCIENCE NO. 225,1975.
C (B) TSCHERNING,C.C. AND R.H.RAPP: CLOSED COVARIANCE EXPRESSIONS
C     FOR GRAVITY ANOMALIES, GEOID UNDULATIONS* AND DEFLECTIONS OF
C     THE VERTICAL IMPLIED BY ANOMALY DEGREE-VARIANCE MODELS. DEP-
C     ARTMENT OF GEODETIC SCIENCE, THE OHIO STATE UNIVERSITY,
C     REPORT NO. 208, 1974.
C
      IMPLICIT REAL *8(A-H,O-Z), LOGICAL (L)
C
      COMMON /CMCOV/CI(12),CR(51),SIGMAO(300),SIGMA(300),HMAX,KI(25),N1,
     *N2,LOCAL,LSUM
C THE COMMON BLOCK CONTAINS THE VALUES OF PARAMETERS USED FOR THE COM-
C PUTATIONS AND RETURN VALUES OF FUNCTIONS AND CONSTANTS, WHICH HAVE
C BEEN USED IN THE COMPUTATIONS.
C PARAMETERS USED FOR THE COMPUTATIONS:
C   CI(8) = THE CUNSTANT A(I) OF REF.(A), EQ.(17) IN UNITS OF (M/SEC)**4
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C    CI(10) THE SQUARE OF THE RATIO BETWEEN THE BJERHAMMAR-SPHERE RADIUS
C    (RB) AND THE MEAN RADIUS OF THE EARTH (RE).
C    CR(1) COSINE OF THE SPHERICAL DISTANCE BETWEEN P AND Q.
C    CR(2),CR(3) THE HEIGHT OF P, Q, RESPECTIVELY, (UNITS METERS),
C    CR(4),CR(5) SINE OF THE LATITUDE OF P, Q, RESPECTIVELY,
C    CR(6),CR(7) COSINE OF THE LATITUDE OF P, Q, RESPECTIVELY,
C    CR(8),CR(9) SINE AND COSINE OF THE LONGITUDE DIFFERENCE,
C    CR(10),CR(11) THE REFERENCE GRAVITY IN P, Q, RESPECTIVELY (WHEN
C    USED, OTHERWISE STORE 1.0D0), (UNITS M/SEC**2).
C    SIGMAO(1)-SIGMAO(N1) MUST CONTAIN THE EMPIRICAL ANOMALY DEGREE-
C    VARIANCES IN UNITS OF MGAL**2.
C    KI(3) = K(2) OF DEG.VAR. MODEL 2 OR 3,
C    KI(4) = K(3) OF DEG.VAR. MODEL 3, CF. REF.(A), EQ.(17).
C    KI(5) = THE DEG.VAR. MODEL NUMBER, (EQUAL TO 1, 2 OR 3),
C    KI(6),KI(7) THE INTEGER SPECIFYING THE KIND OF QUANTITY WHICH IS
C    ASSOCIATED WITH P, Q, RESPECTIVELY,
C    N1 = THE NUMBER OF EMPIRICAL DEGREE-VARIANCES USED (LOCAL =.FALSE.)
C    OR (ORDER+1) OF THE LOCAL COVARIANCE FUNCTION USED (LOCAL=.TRUE.).
C    HMAX, N2, LSUM. HMAX IS THE HEIGHT ABOVE WHICH THE LEGENDRE SERIES
C    OF MAXIMAL DEGREE N2-1 WILL BE USED FOR THE COMPUTATION OF THE CO-
C    VARYANCES WHEN LSUM IS TRUE. N2 MUST BE GREATHER THAN 2 AS WELL AS
C    GREATHER THAN N1.
C RETURN VALUES:
C    CI(1)-CI(7), THE QUANTITIES C(J,Q) OF REF.(A), EQ.(47), WITH
C    CI(1) - CI(KI(5)+1) = C(J,Q), CI(5) = C(KI(5)+2,Q),
C    CI(6) = C(KI(5)+3,Q), CI(7) = C(KI(5)+4,Q),
C    CI(9) = RB**2, CI(11),CI(12) QUANTITIES USED TO GIVE THE COMPUTED
C    COVARIANCES THE PROPER UNITS.
C    CR(ND*8+12), THE VALUES OF THE ND'TH DERIVATIVE OF THE SUM OF THE
C    FINITE LEGENDRE-SERIES, CF.REF.(A), EQ.(20),(48) AND (52).
C    CR(ND*8+13) - CR(ND*8+19), THE VALUES OF THE ND'TH DERIVATIVES OF
C    THE FUNCTIONS F(-2), F(-1), F(KI(3)), F(KI(4)), S0, S1, S2, CF. REF.
C    (A), EQ. (42), (41), (39), (39), (30), (34) AND (35).
C    SIGMAO(1) - SIGMAO(N1) THE POTENTIAL DEGREE-VARIANCE CORRECTIONS*
C    CF. REF.(A), EQ.(16), (AFTER THE CALL OF COVAX).
C    SIGMA(4) - SIGMA(N1), THE POTENTIAL DEGREE-VARIANCES MULTIPLIED BY
C    THE FACTORS GIVEN IN REF.(A), TABLE 1.
C    SIGMA(1) - SIGMA(3), THE DEGREE-VARIANCES OF DEGREE 0,1,2 MINUS
C    TERMS OF THE SAME DEGREES ACQUIRED FROM REF.(A), EQ.(34),(35),(41)
C    AND (42).
C    KI(8),KI(9) THE NUMBER OF DIFFERENTIATIONS IN RADIAL DIRECTION AND
C    WITH RESPECT TO T = COS(SPHERICAL DIST.) TO BE PERFORMED.
C    KI(10) - KI(15) THE CONSTANTS I,K,J,M,J1,M1 OF REF.(A), SECTION 2.
C    KI(16) - KI(19) THE QUANTITIES M(1) - M(4) OF REF.(A), EQ.(26)-(29).
C    KI(20),KI(21) THE EXPONENT OF THE REFERENCE GRAVITY,
C    KI(22),KI(23) THE EXPONENT OF THE RADIAL DISTANCE AND
C    KI(24),KI(25) THE EXPONENT OF COSINE OF THE LATITUDE* OF P, Q RES-
C    PECTIVELY WITH WHICH THEESE QUANTITIES ARE USED IN THE COVARIANCE
C    COMPUTATIONS.
C
      DIMENSION K7(15),K9(15),K11(15),K13(15),K15(15),K17(15),K19(15),
     *K21(15),K23(15),C11(15),K8(15),L(7),LN(7),CX(6,8),SM(N2)
     *,C(6),V(6),U(6),G(6),P(6),R(6),SS1(4),D(36),RM(6),Q(6)
C THE ARRAY SM IS USED TO STORE THE DEGREE-VARIANCES WHEN THE LOGICAL
C VARIADLE LSUM IS TRUE. IN CASE THE SUBSCRIPT LIMIT IS CHANGED IS IT
C NECESSARY TO CHARGE THE VALUE OF THE VARIABLE N2 ACCORDINGLY.
C
      EQUIVALENCE (CX(1,1),C(1)),(CX(1,2),V(1)),(CX(1,3),U(1)),
     *(CX(1,4),G(1)),(CX(1,5),P(1)),(CX(1,6),R(1)),(CX(1,7),SS1(1)),
     *(CX(2,8),SS2)
C
      DATA D0,D1,D2,D3,RE/0.0D0,1.0D0,2.0D0,3.0D0,6371.0D3/
```

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      *,K7/5*0,6*1,3*2,0/,K9/5*1,2,3,2,3,2,3,2,2,3,1/,K11/11*0,2,3,3,0/,
     *K13/11*1,2,3,3,1/,K15/0,1,-1,-1,1,0,0,-1,-1,1,1,0,0,0,0/,
     *K17/3*0,2,2,10*0/,K19/1,4*0,1,1,8*0/,K21/0,2,1,2,2,1,1,7*2,0/,
     *K23/6*0,1,0,1,0,1,0,1,2,0/,K8/0,1,1,2,2,0,0,4*1,4*0/,
     *C11/1.0D0,1.0D9,1.0D5,2*1.0D9,2*-206264.806D0,7*1.0D9,1.0D0/
C THE ARRAYS K7 - K23 CONTAINS TABLES OF QUANTITIES RELATED TO THE KIND
C OF COVARIANCES (1 - 14) WHICH MAY BE COMPUTED. THE ELEMENT WITH SUBS-
C CRIPT 15 IS DUMMY (RESERVED FOR PRGGRAM EXTENSIONS).  THEIR ACTUAL VA-
C LUES WILL AFTER CALL OF COVBX BE STORED IN THE ELEMENTS OF THE ARRAY
C KI HAVING SUBSCRIPTS R - 25.
C K7 CONTAINS THE ORDER OF DIFFFKENTIATION WITH RESPECT TO T,K8 THE
C ORDER OF DIFFERENTIATION WITH RESPECT TO THE RADIUS, GF.REF(A),TABLE
C 1. K9,K11,K13 THE KIND OF DIFFERENTIATIONS TO RE COMPUTED WITH RESPECT
C TO THE LATITUDE (2) AND THE LONGITUDE (3), CF.REF(A),SECTION 3. K15
C AND K17 CONTAINS AN INTEGER, WHICH WILL BE ADDED TCI THE DEGREE,  THE
C SUM WILL THEN RE MULTIPLIED WITH THE DEGREE-VARIANCE OF THE CORRESPON-
C DING DEGREE WHEN A FIRST AND/OR SECOND DIFFERENTIATION HITH RESPECT
C TO THE RADIAL DISTANCE HAS TAKEN PLACE.
C C11 CONTAIN QUANTITIES USED TO GIVE THE COVARIANCES THE PROPER UNITS-
C
      KT = KI(5)
      KT1 = KT+1
      IF (KT.LT.3) GO TO 15
      D0 16 K = KT, 2
   16 KI(K+2) = DO
   15 KI(1) = -2
      KI(2) = -1
C
      IF ((KT.LT.3).OR.(KT.EQ.3.AND.KI(4).GT.KI(3))) GO TO 17
C ASSURING* THAT KI(4).GT.KI(3), BECAUSE THIS FACT IS USED IN SUB-
C SEQUENT COMPUTATIONS.
      K = KI(3)
      KI(3) = KI(4)
      KI(4) = K
   17 II = KI(3)
      JJ = KI(4)
      SM(1) = DO
      SM(2) = DO
      N3 = N1
      A = CI(8)
      S = CI(10)
      RB2 = S*(RE**2)
      CI(9) = RB2
      RB2 = RB2*1.0D-10
      T = DO
      Q(1) = DO
      RM(1) = D0
C
      SIGMAO(1) = DO
      SIGMAO(2) = D0
      IF (LOCAL) SIGMAO(3) = DO
      IF (.NOT.LOCAL) SIGMAO(3) = SIGMAO(3)*RB2/S**4
      IF (N1.LT.4) GO TO 14
      DO 13 K = 4, N1
      IF (.NOT.LOCAL) T = SIGMAO(K)*S**(-K-1)*RB2
      GO TO (10,11,12),KT
   10 KK = 1
      GO TO 13
   11 KK = K+II-1
      GO TO 13
   12 KK = (K+II-1)*(K+JJ-1)
   13 SIGMAO(K) = (T-A*(K-2)/((K-3)*KK))/(K-2)**2
```

```
   14 RETURN
C
C .
      ENTRY COVBX
C BY THE CALL OF COVBX ALL QUANTITIES NECESSARY FOR THE COMPUTATION OF
C THE COVARIANCE, HUT INDEPENDENT OF THE POSITION OF THE POINTS P AND Q,
C ARE COMPUTED.
      RR2 = CI(9)
      CI(11) = D1
      IF (KI(6).EQ.15.OR.KI(7).EQ.15) GO TO 19
C
      DO 20 M = 1, 2
      K = KI(M+5)
C FOR M = 1, K IS EQUAL TO THE KIND EVALUATED IN P AND FOR M = 2 EQUAL
C TO THE KIND FVALUATED IN Q.
      KI(M+9) = K9(K)
      KI(M+11) = K11(K)
      KI(M+13) = K13(K)
      KI(M+15) = K15(K)
      KI(M+17) = K17(K)
      KI(M+19) = K19(K)
      KI(M+21) = K21(K)
      KI(M+23) = K23(K)
C
   20 CI(11) = CI(11)*C11(K)
C
      KQ = K
      KP = KI(6)
      KI(8) = K7(KP)+K7(KQ)
      KI(9) = K8(KP)+K8(KQ)
   19 ND = KI(8)
      NR = KI(9)
C ND AND NR ARE THE NUMBER OF DIFFERENTIATIONS WITH RESPECT TO T AND THE
C RADIAL DISTANCES* RESPECTIVELY.
C
C UPDATING THE DEGREE-VARIANCES, CF. REF(A), TABLE 1.
      SIGMA(1) = DO
      SIGMA(2) = DO
      IF (LSUM) N1 = N2
      DO 21 M = 3, N1
      B = D1
      OO 22 I = 1, 4
   22 IF (KI(I+15).NE.O) R = B*(M+KI(I+15)-1)
      IF (M.LE.N3) SIGMA(M) = SIGMAO(M)*B
      IF (.NOT.LSUM.OR.M.EQ.3) GO TO 21
      DO 48 K = 1, KT1
   48 B = B/(M+KI(K)-1)
C STORING THE MODIFIED DEGREE-VARIANCES OF DEGREE M-I IN SM(M) AND AD-
C OIND THE DEGREE-VARIANCE CORRECTIONS FOR M .LE. N3.
      SM(M) = R+A
      IF (M.LE.N3)  SM(M) = SM(M)+SIGMA(M)
   21 CONTINUE
      IF (N1.GT.2) SM(3) = SIGMA(3)
      IF (LSUM) N1 = N3
C
C EVALUATION OF THE QIJANTITIES C(J,NR), CF.REF(A), TABLE 2.
      DO 23 K = 1, 7
   23 CI(K) = DO
C
      OO 25 K = 1, KTI
      CI(K) = D1
      DO 25 KQ = 1, KT1
```

```
   25 IF (K.NE.KQ) CI(K) = CI(K)/(KI(KQ)-KI(K))
C CF.REF(A),EQ.(19). WE WILL THEN COMPUTE THE QUANTITIES GIVEN IN REF(A)
C TABLE 2.
      IF (NR.LT.2) GO TO 29
      KP = KI(16)+KI(17)+KI(18)+KI(19)
      IF (NR.EQ.4) M = KI(16)*(KI(17)+KI(18)+KI(19))+KI(17)*(KI(18)+
     *KI(19))+KI(18)*KI(19)
C
      GO TO (26,27,28),KT
   26 CI(NR+3) = D1
      IF (NR.GT.2) CI(NR+2) = KP+3
      IF (NR.EQ.4)   CI(NR+1) = M+3*KP+7
      GO TO 29
   27 IF (NR.GT.2) CI(NR+2) = D1
      IF (NR.EQ.4) CI(NR+1) =-KI(3)+3+KP
      GO TO 29
   28 IF (NR.EQ.4) CI(NR+1) = D3
   29 IF (NR.EQ.0) GO TO 31
C
      OO 30 KP = 1, 4
      DO 30 K  = 1, KT1
   30 IF (KI(KP+15).NE.0) CI(K) = CI(K)*(KI(KP+15)-KI(K))
C
C THE LOGICAL ARRAYS L AND LN REGISTER WHICH TERMS THAT WILL HAVE TO
C BE EVALUATFD , RESPECTIVELY NOT EVALUATED IN REF.(A), EQ. (47).
   31 DO 38 K = 1, 7
      L(K) = DABS(CI(K)).GT.1.0D-15
   38 LN(K) = .NOT.(L(K))
C
      OO 32  K = 3, 7
      DO 32  M = 1, 3
      IF (M.EQ.1.AND.K.GT.5.OR.(M+KI(K)-1).EQ.0.AND.K.LT.5.OR.LN(K))
     *GO  TO 32
      GO TO (34,34,35,35,34,36,37),K
   34 B = D1
      GO TO 33
   35 B = D1/(M+KI(K) -1)
      GO TO 33
   36 R = (M- I)
      GO TO 33
   37 B = (M-1)*(M-1)
   33 SIGMA(M) = SIGMA(M)-A*CI(K)*B
   32 CONTINUE
      SIGMA(3) = SIGMA(3)-A*CI(2)
C
      ND1 = ND+1
      ND2 = ND+2
      RETURN
C
      ENTRY COVCX(COV)
C COMPUTATION OF THE COVARIANCE IN A SPECIFIC PAIR OF POINTS.  THE VALUE
C IS RETURNED THROUGH THE PAKAMETFR COV.
C THE COVARIANCES COMPUTED WILL BE IN UNITS CORRESPONDING TO THE KIND OF
C QUANTITIES, I.E. FOR KIND (1) METERS, (2) EOTVOS (E), (3) MGAL,
C (4),(5) E, (6),(7) ARCSECONDS, (8) - (14) E.
C THE FOLLOWING QUANTITIES MUST HE STORED IN THE ELEMENTS OF THE ARRAY
C CR WHEN COVCX IS CALLED: (1) COSINE TO THE SPHERICAL DISTANCE RETHEEN
C P AND Q, (2),(3) THE HEIGHT OF Pc Q RESPECTIVELY, (4),(5) SINE OF THE
C THE LATITUDE OF P, Q, RESPECTIVELY, (6),(7) COSINE OF THE LATITUDE OF
C P, Q, RESPECTIVELY, (8),(9) SINE AND COSINE OF THE LONGITUDE DIFFER-
C ENCE. THF REFERENCE GRAVITY WILL HAVE TO BE STORED IN CR(10),CR(11)
C FOR P, Q RFSPECTIVELY (WHEN USED, OTHFRWISE STORE 1.0).
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```
      T = CR(1)
      HP = CR(2)
      HQ = CR(3)
      SP = CR(4)
      SO = CR(5)
      CP = CR(6)
      CQ = CR(7)
      SD = CR(8)
     'CD = CR(9)
      RP = RE+HP
      RQ = RE+HQ
C IN HEIGH ALTITUDES AND WHEN LSUM IS TRUE WILL THE COVARIANCE BE COM-
C PUTED BY P SUMMATION OF THE LEGENDRE-SERIES ABBREVIATED TO DEGREE
C NZ-1,
      LSUMC = LSUM .AND. (HP.GT.HMAX .OR. HQ.GT.HMAX)
'C COMPUTATION OF THE CONSTANT USED TO CONVERT THE COVAKIANCE INTO
C PRQPER UNITS.
    . CI(12) = CI(11)/(CP**KI(24)*CQ**KI(25)*RP**KI(22)*RQ**KI(23)
     **CR(11)**KI(21)*CR(10)**KI(20))
C
      S = RB2/(RP*RQ)
      S2 = S*S
      ST = S*T
      T2 = T*T
      P2 = (D3*T2-D1)/D2
      P3 = (D3*ST+D1)/D2
C
C INITIALIZING ARRAY ELEMENTS.  NOTE THE USE OF THE EQUIVALENCYNG.
      D0 50  K = 1, H
      DO 50 M = 1, ND2
   50 CX(M,K) = DO
      DO 51 K = 1, ND2
      C(K) = DO
   51 D(K) = DO
      DO 52 K = 1, 40
   52 CR(K+11) = DO
C
C SUMMATION AND DIFFERENTIATION OF THE LEGENDRE SERIES* CF.REF(A),EQ.
C (49) AND (51).
      IF (LSUMC) N1 = N2
      K1 = M1
      K2 = N1+1
      K = N1-1
      DO 54 M = 1, N1
      GI = (D2*K+D1)*S/K1
      GJ = -K1*S2/K2
      K2 = K1
      K1 = K
      K = K-I
      IF (.NOT.LSUMC) SI = SIGMA(K2)
      IF (LSUMC) SI = SM(K2)
      12 = 0
      EI = I
      DO 53 1 = 2, ND2
      B = D(I)
      D(I) = C(I)
      C(I) = GI*(D(I)*T+I2*D(I1))+GJ*B+SI
      SI = DO
      12 = 11
   53 I1 = I
   54 CONTINUE
      IF (LSUMC) N1 = N3
```

```
C
C COMPUTATION OF THE QUANTITIES D(1)-D(36),CF.REF(A),SECTION 3.
      IF (ND.EQ.0) GO TO 55
C
      D(1) = D1
      CS = CP*SQ
      SC = SP*CQ
      SCC = SC*CD
      CC = CP*CQ
      CCS = CC*SD
      CSC = CS*CD
      D(2) = CS-SCC
      D(3) = CCS
      017) = SC-CSC
      0113) = -CCS
      IF (ND.EQ.1) GO TO 55
C
      D(4) = -T
      SCS = SC*SD
      CCC = CC*CD
      SS = SP*SQ
      SSC = SS*CD
      CSS = CS*SD
      D15) = -SCS
      D(6) = -CCC
      D(8) = CC+SSC
      D(9) = -CSS
      D(14) = SCS
      D(15) = CCC
      D(19) = -T
      D(25) = CSS
      D(31) = -CCC
      IF (ND.EQ.2) GO TO 55
C
      SSS = SS*SD
      D(10) = -SC+CSC
      D(11) = SSS
      D(12) = CSC
      D(16) = CCS
      0117) = -SCC
      D(18) = CCS
      D(20) = -CS+SCC
      D(21) = -CCS
      D(26) = -SSS
      D(27) = -CSC
      D(32) = SCC
      D(33) = -CCS
      IF (ND.EQ.3) GO TO 55
C
      D(22) = T
      D(23) = SCS
      D(24) = CCC
      D(28) = -CSS
      D(29) = SSC
      D(30) = -CSS
      Of34) = CCC
      D(35) = SCS
      D(36) = CCC
   55 IF (LSUMC) GO TO 75
C
C COMPUTATION OF THE FUNCTIONS L=R(1), N=1/RN, M=RM(2), FO=P(2), CF.
C REF.(A), EQ. (31)-(33),(40) AND (77A).
```

```
      RL2 = D1-D2*ST+S2
      RL = DSQRT(RL2)
      R(1) = RL
      RL1 = D1/RL
      RN = D1/(D1+RL-ST)
      RL2 = D1/RL2
      RNL = RN*RL1
      RM(2) = D1-RL-ST
      P(2) = S*DLOG(D2*RN)
      RL3 = RL2*RL1
      RL5 = RL3*RL2
      S3 = S2*S
      R(2) = -S*RL1
      IF (ND. EQ.0) GO TO 56
C
C COMPUTATION OF THE 'DERIVATIVES WITH RESPECT TO T.
C CF.  REF.(A), EQ. (77B),(69A),(57).
      R(3) = -S2*RL3
      RM(3) = -R(2)-S
      P(3) = S2*(RNL+RN)
      IF (ND.EQ.1) GO TO 56
C
C CF. REF.(A), EQ. (77C),(69B),(58).
      R(4) = -D3*S3*RL5
      RM(4) = -R(3)
      P(4) = S3*(RL3+(D1+(D2+RL1)*RL1)*RN)*RN
      IF (ND.EQ.2) GO TO 56
C
C CF. REF.(A), EQ. (77D),(69C),(59).
      RL4 = RL2*RL2
      RL7 = RL5*RL2
      S4 = S2*S2
      R(5) = -15.0D0*S4*RL7
      RM(5) = -R(4)
      P(5) = S4*(D3*RL5+((D3+D3*RL1)*RL3+D2*(D1+(D3+(D3+RL1)*RL1)*RL1)
     **RN)*RN)*RN
      IF (ND.EQ.3) GO TO 56
C
C CF.  REF.(A), EQ. (69D),(60).
      S5 = S4*S
      RL6 = RL4*RL2
      RM(6) = -R(5)
      P(6) = D3*S5*((5.0D0*RL7+((4.0D0+5.0D0*RL1)*RL5+((4.0D0+(8.0D0
     *+4.0D0*RL1)*RL1)*RL3+(2.0D0+(8.0D0+(12.0D0+(8.0D0+D2*RL1)*RL1)
     **RL1)*RL1)*RN)*RN)*RN)*RN)
C
   56 IF (LN(2)) GO TO 58
C COMPUTATION OF THE FUNCTION F- I AND ITS DERIVATIVES* CF.  REF.(A),
C EQ. (41) AND (61) - (65).
      U(2) = S*(RM(2)+T*P(2))
      IF (ND2.LT.3) GO TO 58
      DO 57 K = 3, ND2
   57 U(K) = S*(RM(K)+T*P(K)+(K-2)*P(K-1))
C
   58 IF (LN(1)) GO TO 60
C COMPUTATION OF THE FUNCTION F- 2 AND ITS DERIVATIVES* CF.  REF.(A) EQ.
C (42), AND (65)- (68).
      DO 59 K = 2, ND2
      GO TO (61,61,62,63,64,65),K
   61 CY = S*(D1-T2)/4.0D0
      GO TO 59
   62 CY = -ST/D2
```

```
      GO TO 59
   63 CY = D3*P(2)-S/D2
      GO TO 59
   64 CY = 9.0D0*P(3)
      GO TO 59
   65 CY = 18.0D0*P(4)
   59 V(K) = S*(RM(K)*P3+S*((K-2)*D3*RM(K-1)/D2+P2*P(K)+D3*T*P(K-1)*
     *(K-2)+CY))
C
   60 IF (LN(3)) GO TO 73
C COMPUTATION OF THE FUNCTION F1 AND ITS DERIVATIVES, CF. REF.(A) EQ.
C (36), REF. (B), EQ.(101) AND REF. (A), EQ.(70),(71).
      Q(2) = DLOG(D1+D2*S/(D1-S+RL))
      IF (ND.EQ.0) GO TO 66
      Q(3) = S2*RNL
      IF (ND.EQ.1) GO TO 66
      Q(4) = S3*((RL1+D1)*RN+RL2)*RNL
      IF (ND.EQ.2) GO TO 66
      Q(5) = S4*(D3*RL4+((D2+D3*RL1)*RL2+(D2 +(4.0D0+D2*RL1)*RL1)*RN)
     **RN)*RNL
      IF (ND.EQ.3) GO TO 66
      Q(6) = D3*S5*(5.0D0*RL6+((D3+5.0D0*RL1)*RL4+((D2+(6.0D0+4.0D0*RL1)
     **RL1)*RL2+(6.0D0+(6.0D0+D2*RL1)*RL1)*RNL)*RN)*RN)*RNL
C                (2+                        RL1)*RN
C COMPUTATION OF THE FUNCTION F2 AND ITS DERIVATIVES* CF. REF.(A), EQ.
C (3),(72)-(75).
   66 P(2) = (RL-D1+T*Q(2))/S
      IF (ND.EQ.0) GO TO 68
      DO 67  K = 3, ND2
   67 P(K) = (R(K-1)+T*Q(K)+(K-2)*Q(K-1))/S
   68 I1 = II-1
      K1 = 1
      J1 = 11
      IF (I1.GE.2) GO TO 49
      DO 49 M = 2, ND2
      IF (I1.EQ.0) G(M) = Q(M)
      IF (I1.EQ.1) G(M) = P(M)
   49 CONTINUE
      IF (L(4)) J1 = JJ-1
      IF (J1.LE.1) GO TO 71
C
C CF. REF.(A), EQ. (38),(76).
      DO 71  K = 2, J1
      DO 69  M = 2, ND2
      B = Q(M)
      Q(M) = P(M)
   69 P(M) = (R(M-1)+(2*K-1)*((M-2)*Q(M-1)+T*Q(M))-K1/S*B)/(K*S)
      IF (K.NE.I1) GO TO 71
      DO 70 M = 2, ND2
   70 G(M) = P(M)
   71 K1 = K
C
   73 IF (LN(6)) GO TO 72
C CF. REF.(A), EQ. (34),(55).
      SS1(2) = S2*(T-S)*RL3
      IF (ND.GT.0) SS1(3) = S2*(RL3+D3*(T-S)*S*RL5)
C
C CF. REF.(A), EQ. (35).
   72 IF (L(7)) SS2= S2*((T+S)*RL3+D3*S*(T2-D1)*RL5)
C
C ADDING THE DIFFERENT TERMS, CF. REF.(A), EQ. (22),(47).
C TIPLIED BY RB**2 IN UNITS OF MGAL**2, THE INTEGERS K(2),K(3) OF EQ.
```

```
   75 DO 78 M = 2, ND2
C CF.  REF.(A), EQ. (50),(52).
      C(M) = S*C(M)
      CR(M*8 -4) = C(M)
      DO 78 K = 1, 7
      IF (LN(K)) GO TO 78
C STORING THE TERMS FOR TRANSFER TO THE CALLING PROGRAM USING THE COMMON
C AREA /CMCOV/.                                M*8+K-4)
      CR(M*8+K -4) = A*CX(M,K+1)*CI(K)
      IF (K. EQ.5) CR(M*8+K-4) = -CR(
      C(M) = C(M)+CR(M*8+K -4)
   78 CONTINUE
C
C INTEGERS SPECIFYING THE KINDS OF DIFFERENTIATION WITH RESPECT TO THE
C LATITUDES AND/OR THE LONGITUDES, CF. REF.(A), SECTION 3.
      I = KI(10)
      J = KI(12)
      K = KI(11)
      M = KI(13)
      J1 = KI(14)
      M1 = KI(15)
C
C COMPUTATION OF THE DERIVATIVES OF ORDER ND WITH RESPECT TO THE LATI-
C TUDES AND THE LONGITUDES, CF. REF.(A), EQ. (43) - (46).
      GO TO (80,81,82,83,84),ND1
   80 COV = C(2)
      GO TO 85
   81 COV = C(3)*D(I+6*(K-1))
      GO TO 85
   82 COV = D(I)*D(J1)*D(6*(K-1)+1)*D(6*(M1-1)+1)*C(4)+D(I+J+6*(K+M-1))
     **C(3)
      GO TO 85
   83 COV = D(I+J+6*(K+M-1))*C(3)+(D(I+J)*D(6*(K+M-1)+1)+D(I+6*(K-1))
     **D(J1+6*(M1-1))+D(I+6*(M1-1))*D(J1+6*(K-1)))*C(4)
     *+D(I)*D(J1)*D(6*(K-1)+1)*D(6*(M1-1)+1)*C(5))
      GO TO 85
   84 COV  = D(I+J+6*(K+M-1))*C(3)+(D(I+J+6*(K-1))*D(6*(M-1)+1)
     *+D(I+6*(M-1+K))*D(J)+D(J+6*(K+M-1))*D(I)+D(I+J+6*(M-1))
     **D((K-1)*6+1)+D(I+J)*D(6*(K+M-1)+1)+D(I+6*(K-1))*D(J+6*(M-1))
     *+D(I+6*(M-1))*D(J+6*(K-1)))*C(4)+(D(I+J)*D(6*(K-1)+1)*D(6*(M-1)+1)
     *+D(I+6*(K-1))*D(J)*D(6*(M-1)+1)+D(I+6*(M-1))*D(J)*D(6*(K-1)+1)
     *+D(J+6*(K-1))*D(I)*D(6*(M-1)+1)+D(J+6*(M-1))*D(I)*D(6*(K-1)+1)
     *+D(6*(K+M-1)+1)*D(I)*D(J))*C(5)+D(I)*D(J)*D(6*(K-1)+1)*D(6*(M-1)
     *+1)*C(6)
C
C GIVING THE COVARIANCE THE PROPER UNITS.
   85 COV = COV*CI(12)
C
      RETURN
      END
```

~TEST

Sample Input Data

```
F
 .999617 D  0 4.2528   D  2   24    0      2    2 F    0  .0      D  0 T
 0.00  0.00  7.50
   4  30.00    0 00 00.00 F
 1  1       0       0 F  2  1      0      0 F  5  1      0      0 F
12  1       0       0 F 14  1      0      0 T
   4  30.00    0 00 00.00 T
 3  3    1000    1000 F  2  3   1000   1000 F  5  3   1000   1000 F
12  3    1000    1000 F 14  3   1000   1000 T
```

Output example:

TEST OF COVARIANCE FUNCTION SUBROUTINES, VERS. JUNE 75.

COVARIANCES BETWEEN QUANTITIES OF KIND KP,KQ ARE COMPUTED,
THE KINDS AND CORRESPONDING UNITS ARE AS FOLLOWS: (E=EOTVOS):
(1) THE HEIGHT ANOMALY (METERS), (2) THE NEGATIVE RADIAL DER-
IVATIVE DIVIDED BY THE RADIAL DISTANCE (E), (3) THE GRAVITY
ANOMALY (MGAL), (4) THE RADIAL DERIVATIVE OF (3) (E), (5) THE
SECOND ORDER RADIAL DERIVATIVE (E), (6),(7) THE LATITUDE AND
THE LONGITUDE COMPONENTS OF THE DEFLECTIONS OF THE VERTICAL
(ARCSECONDS), (8),(9) THE DERIVATIVES OF (3) IN NORTHERN AND
EASTERN DIRECTION, RESPECTIVELY (E), (10),(11) THE DERIVATIVE
OF (2) IN THE SAME DIRECTIONS (E), (12)-(14) THE SECOND ORDER
DERIVATIVES IN NORTHERN, (NORTHERN,EASTERN) AND EASTERN
DIRECTIONS, RESPECTIVELY (E).

PARAMETERS SPECIFYING THE MODEL DEGREE-VARIANCES:
S,A = 0.9996170D+00 0.4252800D+03
KO-K3,N,KT= -2 -1 24 0 2 2

EMPIRICAL ANOMALY DEGREE-VARTANCES IN UNITS OF MGAL**2:
0.0 0.0 7.50

MODEL DEGREE-VARIANCE CORRECTIONS:
0.0 0.0 0.3048D+05

TABLE OF COVARIANCES:
BETWEEN QUANTITIES OF KIND KP AND KQ, EVALUATED IN P,Q,
HAVING SPHERICAL DISTANCE PSI, HEIGHTS HP, HQ
AND AN AZIMUTH OF 0 D 0 M 0.0 SEC FROM P TO Q.

	KP=	1	2	5	12	14
	KQ=	1	1	1	1	1
	HP=	0.0	0.0	0.0	0.0	0.0
	HQ=	0.0	0.0	0.0	0.0	0.0
PSI						
0	0.0	926.59371	1.15780	23.19983	-10.44212	-10.44212
0	30.00	925.47496	1.12971	12.93102	-4.43459	-6.23701
1	0.0	922.88515	1.10161	10.45730	-3.25429	-4.99980
1	30.00	919.25573	1.07576	9.10668	-2.64768	-4.30748

TABLE OF COVARIANCES:
BETWEEN QUANTITIES OF KIND KP AND KQ, EVALUATED TN P,Q,
HAVING SPHERICAL DISTANCE PSI, HEIGHTS HP, HQ
AND AN AZIMUTH OF 0 D 0 M 0.0 SEC FROM P TO Q.

	KP=	3	2	5	12	14
	KQ=	3	3	3	3	3
	HP=	1000.0	1000.0	1000.0	1000.0	1000.0
	HQ=	1000.0	1000.0	1000.0	1000.0	1000.0
PSI						
0	0.0	1551.47275	2.65154	911.69904	-453.19798	-453.19798
0	30.00	791.63914	1.45146	53.79638	4.37320	-55.26665
1	0.0	568.76144	1.09365	22.32984	4.95074	-25.09328
1	30.00	450.43329	0.90066	13.03381	4.00347	-15.23596